43. Let $w = \sqrt{x}$. Then $dw = \dfrac{dx}{2\sqrt{x}}$, so

$$dx = 2\sqrt{x}\ dw = 2w\,dw.$$

$$\int \sin \sqrt{x}\ dx = \int (\sin w)(2w\ dw)$$
$$= 2\int w \sin w\ dw$$

Let $u = w$ $\quad dv = \sin w\ dw$
$\quad du = dw$ $\quad v = -\cos w$

$$\int w \sin w\ dw = -w \cos w + \int \cos w\ dw$$
$$= -w \cos w + \sin w + C$$

$$\int \sin\sqrt{x}\ dx = 2\int w \sin w\ dw$$
$$= -2w \cos w + 2 \sin w + C$$
$$= -2\sqrt{x}\ \cos\sqrt{x} + 2 \sin\sqrt{x} + C$$

44. Let $w = \sqrt{3x+9}$. Then $dw = \dfrac{1}{2\sqrt{3x+9}}(3)\ dx$,

so $dx = \dfrac{2}{3}\sqrt{3x+9}\ dw = \dfrac{2}{3} w\ dw$.

$$\int e^{\sqrt{3x+9}}\ dw = \int (e^w)\left(\frac{2}{3} w\ dw\right) = \frac{2}{3}\int w\, e^w\ dw$$

Let $u = w$ $\quad dv = e^w\ dw$
$\quad du = dw$ $\quad v = e^w$

$$\int w\, e^w\ dw = w\, e^w - \int e^w\ dw$$
$$= w\, e^w - e^w$$
$$= (w-1)e^w$$

$$\int e^{\sqrt{3x+9}} dx = \frac{2}{3}\int w\, e^w\ dw$$
$$= \frac{2}{3}(w-1)e^w$$
$$= \frac{2}{3}\left(\sqrt{3x+9} - 1\right)e^{\sqrt{3x+9}} + C$$

45. Let $w = x^2$. Then $dw = 2x\ dx$.

$$\int x^7 e^{x^2} dx = \int (x^2)^3 e^{x^2}\ x\ dx = \frac{1}{2}\int w^3\ e^w\ dw.$$

Use tabular integration with $f(x) = w^3$ and $g(w) = e^w$.

$f(w)$ and its derivatives		$g(w)$ and its integrals
w^3	(+)	e^w
$3w^2$	(−)	e^w
$6w$	(+)	e^w
6	(−)	e^w
0		e^w

$$\int w^3\, e^w\ dw$$
$$= w^3\, e^w - 3w^2 e^w + 6w\, e^w - 6e^w + C$$
$$= (w^3 - 3w^2 + 6w - 6)\, e^w + C$$

$$\int x^7 e^{x^2}\ d = \frac{1}{2}\int w^3\ e^w\ dw$$
$$= \frac{1}{2}(w^3 - 3w^2 + 6w - 6)e^w + C$$
$$= \frac{(x^6 - 3x^4 + 6x^2 - 6)e^{x^2}}{2} + C$$

46. Let $y = \ln r$. Then $dy = \dfrac{1}{r} dr$, and so $dr = r\ dy = e^y\ dy$. Using the result of Exercise 17, we have:

$$\int \sin\ (\ln r)\ dr$$
$$= \int (\sin y) e^y\ dy$$
$$= \frac{1}{2} e^y (\sin y - \cos y) + C$$
$$= \frac{1}{2} e^{\ln r}[\sin\ (\ln r) - \cos\ (\ln r)] + C$$
$$= \frac{r}{2}[\sin\ (\ln r) - \cos\ (\ln r)] + C$$

47. Let $u = x^n$ $\quad dv = \cos x\ dx$
$\quad du = nx^{n-1}dx$ $\quad v = \sin x$

$$\int x^n \cos x\ dx = x^n \sin x - \int (\sin x)(nx^{n-1}dx)$$
$$= x^n \sin x - n\int x^{n-1} \sin x\ dx$$

48. Let $u = x^n$ $\quad dv = \sin x\ dx$
$\quad du = nx^{n-1}dx$ $\quad v = -\cos x$

$$\int n^x \sin x\ dx$$
$$= (x^n)(-\cos x) - \int (-\cos x)(nx^{n-1})\ dx$$
$$= -x^n \cos x + n\int x^{n-1} \cos x\ dx$$

49. Let $u = x^n \qquad dv = e^{ax}\,dx$

$du = nx^{n-1}\,dx \qquad v = \frac{1}{a}e^{ax}$

$$\int x^n e^{ax}\,dx$$
$$= (x^n)\left(\frac{1}{a}e^{ax}\right) - \int\left(\frac{1}{a}e^{ax}\right)(nx^{n-1}\,dx)$$
$$= \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx,\ a \neq 0$$

50. Let $u = (\ln x)^n \qquad dv = dx$

$du = \frac{n(\ln x)^{n-1}}{x}\,dx \qquad v = x$

$$\int (\ln x)^n\,dx = (\ln x)^n(x) - \int x\left[\frac{n(\ln x)^{n-1}}{x}\right]dx$$
$$= x(\ln x)^n - n\int(\ln x)^{n-1}\,dx$$

51. **(a)** Let $y = f^{-1}(x)$. Then $x = f(y)$, so $dx = f'(y)\,dy$. Hence,

$$\int f^{-1}(x)\,dx = \int (y)[f'(y)\,dy]$$
$$= \int y\,f'(y)\,dy$$

(b) Let $u = y \qquad dv = f'(y)dy$

$du = dy \qquad v = f(y)$

$$\int y\,f'(y)\,dy = y\,f(y) - \int f(y)\,dy$$
$$= f^{-1}(x)(x) - \int f(y)\,dy$$

Hence,

$$\int f^{-1}(x)\,dx = \int y\,f'(y)\,dy$$
$$= x\,f^{-1}(x) - \int f(y)\,dy.$$

52. Let $u = f^{-1}(x) \qquad dv = dx$

$du = \left(\frac{d}{dx}f^{-1}(x)\right)dx \qquad v = x$

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx$$

53. **(a)** Using $y = f^{-1}(x) = \sin^{-1}x$ and $f(y) = \sin y,\ -\frac{\pi}{2} \le y \le \frac{\pi}{2}$, we have:

$$\int \sin^{-1}x\,dx = x\sin^{-1}x - \int \sin y\,dy$$
$$= x\sin^{-1}x + \cos y + C$$
$$= x\sin^{-1}x + \cos(\sin^{-1}x) + C$$

(b) $\int \sin^{-1}x\,dx$

$$= x\sin^{-1}x - \int x\left(\frac{d}{dx}\sin^{-1}x\right)dx$$
$$= x\sin^{-1}x - \int x\frac{1}{\sqrt{1-x^2}}dx$$

$u = 1 - x^2,\ du = -2x\,dx$

$$= x\sin^{-1}x + \frac{1}{2}\int u^{-1/2}du$$
$$= x\sin^{-1}x + u^{1/2} + C$$
$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$

(c) $\cos(\sin^{-1}x) = \sqrt{1-x^2}$

54. **(a)** Using $y = f^{-1}(x) = \tan^{-1}$ and $f(y) = \tan y,\ -\frac{\pi}{2} < y < \frac{\pi}{2}$, we have:

$$\int \tan^{-1}x\,dx$$
$$= x\tan^{-1}x - \int \tan y\,dy$$
$$= x\tan^{-1}x - \ln|\sec y| + C$$
$$= x\tan^{-1}x + \ln|\cos y| + C$$
$$= x\tan^{-1}x + \ln|\cos(\tan^{-1}x)| + C$$

(b) $\int \tan^{-1}x\,dx$

$$= x\tan^{-1}x - \int x\left(\frac{d}{dx}\tan^{-1}x\right)dx$$
$$= x\tan^{-1}x - \int x\left(\frac{1}{1+x^2}\right)dx$$

$u = 1 + x^2,\ du = 2x\,dx$

$$= x\tan^{-1}x - \frac{1}{2}\int u^{-1}\,du$$
$$= x\tan^{-1}x - \frac{1}{2}\ln|u| + C$$
$$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$$

(c) $\ln|\cos(\tan^{-1}x)| = \ln\left|\frac{1}{\sqrt{1+x^2}}\right|$

$$= -\frac{1}{2}\ln(1+x^2)$$

55. **(a)** Using $y = f^{-1}(x) = \cos^{-1} x$ and $f(y) = \cos x$, $0 \le x \le \pi$, we have:

$$\int \cos^{-1} x\, dx = x\cos^{-1} x - \int \cos y\, dy$$
$$= x\cos^{-1} x - \sin y + C$$
$$= x\cos^{-1} x - \sin\,(\cos^{-1} x) + C$$

(b) $\int \cos^{-1} x\, dx$

$$= x\cos^{-1} x - \int x\left(\frac{d}{dx}\cos^{-1} x\right) dx$$
$$= x\cos^{-1} x - \int x\left(-\frac{1}{\sqrt{1-x^2}}\right) dx$$

$u = 1 - x^2$, $du = -2x\, dx$

$$= x\cos^{-1} x - \frac{1}{2}\int u^{-1/2}\, du$$
$$= x\cos^{-1} x - u^{1/2} + C$$
$$= x\cos^{-1} x - \sqrt{1-x^2} + C$$

(c) $\sin\,(\cos^{-1} x) = \sqrt{1-x^2}$

56. **(a)** Using $y = f^{-1}(x) = \log_2 x$ and $f(y) = 2^y$, we have

$$\int \log_2 x\, dx = x\log_2 x - \int 2^y\, dy$$
$$= x\log_2 x - \frac{2^y}{\ln 2} + C$$
$$= x\log_2 x - \frac{1}{\ln 2} 2^{\log_2 x}$$

(b) $$\int \log_2 x\, dx = x\log_2 x - \int x\left(\frac{d}{dx}\log_2 x\right) dx$$
$$= x\log_2 x - \int x\left(\frac{1}{x\ln 2}\right) dx$$
$$= x\log_2 x - \int \frac{dx}{\ln 2}$$
$$= x\log_2 x - \left(\frac{1}{\ln 2}\right) + C$$

(c) $2^{\log_2 x} = x$

57. Let $u = \sec x$ $\quad dv = \sec^2 x\, dx$

$du = \sec x \tan x\, dx$ $\quad v = \tan x$

$$\int \sec^3 x\, dx$$
$$= \sec x \tan x - \int \sec x \tan^2 x\, dx$$
$$= \sec x \tan x - \int \sec x(\sec^2 x - 1)dx$$
$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$
$$\int \sec^3 x\, dx$$
$$= \sec x \tan x - \int \sec^3 x\, dx + \ln|\sec x + \tan x|$$

Add $\int \sec^3 x\, dx$ to both sides.

$$2\int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x|$$

Multiply both sides by $\frac{1}{2}$.

$$\int \sec^3 x\, dx$$
$$= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$$

58. Let $u = \csc x$ $\quad dv = \csc^2 x\, dx$

$du = -\csc x \cot x\, dx$ $\quad v = -\cot x$

$$\int \csc^2 x\, dx$$
$$= -\csc x \cot x - \int \csc x \cot^2 x\, dx$$
$$= -\csc x \cot x - \int \csc x(\csc^2 x - 1)dx$$
$$= -\csc x \cot x - \int \csc^3 x\, dx + \int \csc x\, dx$$
$$\int \csc^3 x\, dx$$
$$= -\csc x \cot x - \int \csc^3 x\, dx - \ln|\csc x + \cot x|$$

Add $\int \csc^3 x\, dx$ to both sides.

$$2\int \csc^3 x\, dx = -\csc x \cot x - \ln|\csc x + \cot x|$$

Multiply both sides by $\frac{1}{2}$.

$$\int \csc^3 x\, dx$$
$$= -\frac{1}{2}(\csc x \cot x + \ln|\csc x + \cot x|) + C$$

Quick Quiz Sections 7.1–7.3

1. E

2. C; $\sqrt{x} = \sin y$; $x = \sin^2 y$; $dx = 2\sin y \cos y\, dy$

$$x = 0 \Rightarrow \sin y = \sqrt{0} \Rightarrow y = 0$$

$$x = \frac{1}{2} \Rightarrow \sin y = \frac{\sqrt{2}}{2} \Rightarrow y = \frac{\pi}{4}$$

$$\int_0^{1/2} \frac{\sqrt{x}}{\sqrt{1-x}}\, dx$$

$$= 2\int_0^{\pi/4} \frac{\sin y}{\sqrt{1-\sin^2 y}} \sin y \cos y\, dy$$

$$= 2\int_0^{\pi/4} \sin^2 y\, dy$$

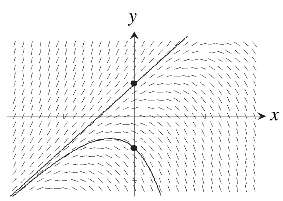
3. A; $\int xe^{2x}dx$

$dv = e^{2x}dx \qquad v = \int e^{2x}dx = \frac{e^{2x}}{2}$

$u = x \qquad du = dx$

$$\frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

4. **(a)**



(b) Let $\frac{dy}{dx} = 2$ and $y = 2x + b$ in the differential equation:

$$2 = 2(2x+b) - 4x$$
$$2 = 2b$$
$$b = 1$$

(c) First, note that $\frac{dy}{dx} = 2(0) - 4(0) = 0$ at the point (0, 0).

Also, $\frac{d^2y}{dx^2} = \frac{d}{dx}(2y - 4x) = 2\frac{dy}{dx} - 4$,
which is –4 at the point (0, 0).
By the Second Derivative test, g has a local maximum at (0, 0).

Section 7.4 Exponential Growth and Decay (pp. 354–365)

Exploration 1 Choosing a Convenient Base

1. $2y_0 = y_0 2^{h\cdot 5}$

$$2 = 2^{5h}$$
$$5h = 1$$
$$h = \frac{1}{5},$$ h is the reciprocal of the doubling period.

2.

$$3 = 2^{\frac{1}{5}t}$$
$$\log 3 = \frac{1}{5}t \log 2$$
$$\frac{5\log 3}{\log 2} = t = 7.925 \text{ years.}$$

3. $3y_0 = y_0 3^{h\cdot 10}$

$$3^1 = 3^{10h}$$
$$h = \frac{1}{10},$$

h is the reciprocal of the tripling period.

4.

$$2 = 3^{\frac{1}{10}t}$$
$$\log 2 = \frac{1}{10}t \log 3$$
$$\frac{10\log 2}{\log 3} = t = 6.3093 \text{ years.}$$

5. $\frac{1}{2}y_0 = y_0\left(\frac{1}{2}\right)^{h\cdot 15}$

$$\left(\frac{1}{2}\right)^1 = \left(\frac{1}{2}\right)^{15h}$$
$$h = \frac{1}{15};$$

h is the reciprocal of the half-life.

6.

$$.10 = \left(\frac{1}{2}\right)^{\frac{1}{15}t}$$
$$\log(0.10) = \frac{1}{15}t \log\left(\frac{1}{2}\right)$$
$$\frac{15\log(0.10)}{\log\left(\frac{1}{2}\right)} = t = 49.83 \text{ years.}$$

Quick Review 7.4

1. $a = e^b$

2. $c = \ln d$

3. $\ln(x+3) = 2$
$$x+3 = e^2$$
$$x = e^2 - 3$$

4. $100e^{2x} = 600$
$$e^{2x} = 6$$
$$2x = \ln 6$$
$$x = \frac{1}{2}\ln 6$$

5. $0.85^x = 2.5$
$$\ln 0.85^x = \ln 2.5$$
$$x \ln 0.85 = \ln 2.5$$
$$x = \frac{\ln 2.5}{\ln 0.85}$$

6. $2^{k+1} = 3^k$
$$\ln 2^{k+1} = \ln 3^k$$
$$(k+1)\ln 2 = k\ln 3$$
$$\ln 2 = k(\ln 3 - \ln 2)$$
$$k = \frac{\ln 2}{\ln 3 - \ln 2}$$

7. $1.1^t = 10$
$$\ln 1.1^t = \ln 10$$
$$t \ln 1.1 = \ln 10$$
$$t = \frac{\ln 10}{\ln 1.1} = \frac{1}{\log 1.1}$$

8. $e^{-2t} = \frac{1}{4}$
$$-2t = \ln\left(\frac{1}{4}\right)$$
$$t = -\frac{1}{2}\ln\left(\frac{1}{4}\right) = \frac{1}{2}\ln 4 = \ln 2$$

9. $\ln(y+1) = 2x-3$
$$y+1 = e^{2x-3}$$
$$y = -1 + e^{2x-3}$$

10. $\ln|y+2| = 3t-1$
$$|y+2| = e^{3t-1}$$
$$y+2 = \pm e^{3t-1}$$
$$y = -2 \pm e^{3t-1}$$

Section 7.4 Exercises

1. $\int y\,dy = \int x\,dx$
$$\frac{y^2}{2} = \frac{x^2}{2} + C$$
$$(2)^2 = (1)^2 + C$$
$$C = 3$$
$y = \sqrt{x^2+3}$, valid for all real numbers

2. $\int y\,dy = -\int x\,dx$
$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$
$$(3)^2 = -(4)^2 + C$$
$$C = 25$$
$y = \sqrt{25 - x^2}$, valid on the interval $(-5, 5)$

3. $\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$
$$\ln|y| = \ln|x| + C$$
$$|y| = |x| + C$$
$$2 = 2 + C$$
$$C = 0$$
$y = x$, valid on the interval $(0, \infty)$

4. $\int \frac{1}{y}\,dy = \int 2x\,dx$
$$\ln y = x^2 + C$$
$$|y| = e^{x^2+C} = e^C e^{x^2}$$
$$y = Ae^{x^2}$$
$$3 = Ae^{0^2}$$
$$3 = A$$
$y = 3e^{x^2}$, valid for all real numbers

5. $\int \frac{dy}{y+5} = \int (x+2)\,dx$
$$\ln|y+5| = \frac{x^2}{2} + 2x + C$$
$$|y+5| = e^{x^2/2+2x+C} = e^C e^{x^2/2+2x}$$
$$y+5 = \pm e^C e^{x^2/2+2x} = Ae^{x^2/2+2x}$$
$$y = Ae^{x^2/2+2x} - 5$$
$$y = 6e^{x^2/2+2x} - 5,$$
valid for all real numbers.

6. $\int \frac{dy}{\cos^2 y} = \int dx$

$\tan y = x + C$

$\tan(0) = 0 + C$

$C = 0$

$y = \tan^{-1} x$, valid for all real numbers.

7. $\frac{dy}{dx} = \cos x\, e^y e^{\sin x}$

$\int e^{-y}\, dy = \int \cos x\, e^{\sin x}\, dx$

$-e^{-y} = e^{\sin x} + C$

$-e^0 = e^{\sin 0} + C$

$C = -2$

$y = -\ln(2e - e^{\sin x})$, valid for all real numbers.

8. $\frac{dy}{dx} = e^{-y} e^x$

$\int e^y\, dy = \int e^x\, dx$

$e^y = e^x + C$

$C = e^2 - e^0 = e^2 - 1$

$y = \ln(e^x + e^2 - 1)$, valid for all real numbers.

9. $\int \frac{1}{y^2}\, dy = \int -2x\, dx$

$-y^{-1} = -x^2 + C$

$\frac{1}{.25} = 1 + C$

$C = 3$

$y = \frac{1}{x^2 + 3}$, valid for all real numbers.

10. $\frac{dy}{dx} = \frac{4\sqrt{y}\ \ln x}{x}$

$\int \frac{dy}{\sqrt{y}} = \int \frac{4 \ln x}{x}\, dx$

$u = \ln x$

$du = \frac{1}{x}\, dx$

$2\sqrt{y} = \int 4u\, du$

$2\sqrt{y} = 2u^2 + C$

$y = (\ln x)^4 + C$

$1 = (\ln e) + C$

$C = 0$

$y = (\ln x)^4$, valid on the interval $(0, \infty)$.

11. $y(t) = y_0 e^{kt}$

$y(t) = 100e^{1.5t}$

12. $y(t) = y_0 e^{kt}$

$y(t) = 200e^{-0.5t}$

13. $y(t) = y_0 e^{kt}$

$y(t) = 50e^{kt}$

$y(5) = 100 = 50e^{5k}$

$2 = e^{5k}$

$\ln 2 = 5k$

$k = 0.2 \ln 2$

Solution: $y(t) = 50e^{(0.2 \ln 2)t}$ or

$y(t) = 50 \cdot 2^{0.2t}$

14. $y(t) = y_0 e^{kt}$

$y(t) = 60e^{kt}$

$y(10) = 30 = 60e^{10k}$

$\frac{1}{2} = e^{10k}$

$\ln \frac{1}{2} = 10k$

$k = 0.1 \ln \frac{1}{2} = -0.1 \ln 2$

Solution: $y(t) = 60e^{-(0.1 \ln 2)t}$ or

$y(t) = 60 \cdot 2^{-t/10}$

15. Doubling time:

$A(t) = A_0 e^{rt}$

$2000 = 1000e^{0.086t}$

$2 = e^{0.086t}$

$\ln 2 = 0.086t$

$t = \frac{\ln 2}{0.086} \approx 8.06$ yr

Amount in 30 years:

$A = 1000e^{(0.086)(30)} \approx \$13,197.14$

16. Annual rate:

$A(t) = A_0 e^{rt}$

$4000 = 2000e^{(r)(15)}$

$2 = e^{15r}$

$\ln 2 = 15r$

$r = \frac{\ln 2}{15} \approx 0.0462 = 4.62\%$

Amount in 30 years:

$$A(t) = A_0 e^{rt}$$
$$A = 2000e^{[(\ln 2)/15](30)}$$
$$= 2000e^{2 \ln 2}$$
$$= 2000 \cdot 2^2$$
$$= \$8000$$

17. Initial deposit:

$$A(t) = A_0 e^{rt}$$
$$2898.44 = A_0 e^{(0.0525)(30)}$$
$$A_0 = \frac{2898.44}{e^{1.575}} \approx \$600.00$$

Doubling time:

$$A(t) = A_0 e^{rt}$$
$$1200 = 600e^{0.0525t}$$
$$2 = e^{0.0525t}$$
$$\ln 2 = 0.0525t$$
$$t = \frac{\ln 2}{0.0525} \approx 13.2 \text{ years}$$

18. Annual rate:

$$A(t) = A_0 e^{rt}$$
$$10{,}405.37 = 1200e^{(r)(30)}$$
$$\frac{104.0537}{12} = e^{30r}$$
$$\ln \frac{104.0537}{12} = 30r$$
$$r = \frac{1}{30} \ln \frac{104.0537}{12} \approx 0.072 = 7.2\%$$

Doubling time:

$$A(t) = A_0 e^{rt}$$
$$2400 = 1200e^{0.072t}$$
$$2 = e^{0.072t}$$
$$\ln 2 = 0.072t$$
$$t = \frac{\ln 2}{0.072} \approx 9.63 \text{ years}$$

19. **(a)** Annually:

$$2 = 1.0475^t$$
$$\ln 2 = t \ln 1.0475$$
$$t = \frac{\ln 2}{\ln 1.0475} \approx 14.94 \text{ years}$$

(b) Monthly:

$$2 = \left(1 + \frac{0.0475}{12}\right)^{12t}$$
$$\ln 2 = 12t \ln \left(1 + \frac{0.0475}{12}\right)$$
$$t = \frac{\ln 2}{12 \ln \left(1 + \frac{0.0475}{12}\right)} \approx 14.62 \text{ years}$$

(c) Quarterly:

$$2 = \left(1 + \frac{0.0475}{4}\right)^{4t}$$
$$\ln 2 = 4t \ln 1.011875$$
$$t = \frac{\ln 2}{4 \ln 1.011875} \approx 14.68 \text{ years}$$

(d) Continuously:

$$2 = e^{0.0475t}$$
$$\ln 2 = 0.0475t$$
$$t = \frac{\ln 2}{0.0475} \approx 14.59 \text{ years}$$

20. **(a)** Annually:

$$2 = 1.0825^t$$
$$\ln 2 = t \ln 1.0825$$
$$t = \frac{\ln 2}{\ln 1.0825} \approx 8.74 \text{ years}$$

(b) Monthly:

$$2 = \left(1 + \frac{0.0825}{12}\right)^{12t}$$
$$\ln 2 = 12t \ln \left(1 + \frac{0.0825}{12}\right)$$
$$t = \frac{\ln 2}{12 \ln \left(1 + \frac{0.0825}{12}\right)} \approx 8.43 \text{ years}$$

(c) Quarterly:

$$2 = \left(1 + \frac{0.0825}{4}\right)^{4t}$$
$$\ln 2 = 4t \ln 1.020625$$
$$t = \frac{\ln 2}{4 \ln 1.020625} \approx 8.49 \text{ years}$$

(d) Continuously:

$$2 = e^{0.0825t}$$
$$\ln 2 = 0.0825t$$
$$t = \frac{\ln 2}{0.0825} \approx 8.40 \text{ years}$$

21. $\frac{dy}{dt} = -0.0077y$

$$\int \frac{1}{y}\,dy = \int -0.0077\,dt$$

$$\ln y = -0.0077t$$

$$t = \frac{\ln\left(\frac{1}{2}\right)}{-0.0077} = 90 \text{ years}$$

22. $\frac{dy}{dt} = -ky$

$$\int \frac{1}{y}\,dy = \int (-k)\,dt$$

$$\ln y = -kt$$

$$-\frac{\ln\left(\frac{1}{2}\right)}{65} = k$$

$$k = 0.01067$$

23. **(a)** Since there are 48 half-hour doubling times in 24 hours, there will be $2^{48} \approx 2.8 \times 10^{14}$ bacteria.

(b) The bacteria reproduce fast enough that even if many are destroyed there are still enough left to make the person sick.

24. Using $y = y_0 e^{kt}$, we have $10{,}000 = y_0 e^{3k}$ and $40{,}000 = y_0 e^{5k}$. Hence $\frac{40{,}000}{10{,}000} = \frac{y_0 e^{5k}}{y_0 e^{3k}}$, which gives $e^{2k} = 4$, or $k = \ln 2$. Solving $10{,}000 = y_0 e^{3\ln 2}$, we have $y_0 = 1250$. There were 1250 bacteria initially. We could solve this more quickly by noticing that the population increased by a factor of 4, i.e., doubled twice, in 2 hrs, so the doubling time is 1 hr. Thus in 3 hrs the population would have doubled 3 times, so the initial population was $\frac{10{,}000}{2^3} = 1250.$

25.

$$0.9 = e^{-0.18t}$$

$$\ln 0.9 = -0.18t$$

$$t = -\frac{\ln 0.9}{0.18} \approx 0.585 \text{ day}$$

26. **(a)** Half-life $= \frac{\ln 2}{k} = \frac{\ln 2}{0.005} \approx 138.6$ days

(b)

$$0.05 = e^{-0.005t}$$

$$\ln 0.05 = -0.005t$$

$$t = -\frac{\ln 0.05}{0.005} \approx 599.15 \text{ days}$$

The sample will be useful for about 599 days.

27. Since $y_0 = y(0) = 2$, we have:

$$y = 2e^{kt}$$

$$5 = 2e^{(k)(2)}$$

$$\ln 5 = \ln 2 + 2k$$

$$k = \frac{\ln 5 - \ln 2}{2} = 0.5\ln 2.5$$

Function: $y = 2e^{(0.5\ln 2.5)t}$ or $y \approx 2e^{0.4581t}$

28. Since $y_0 = y(0) = 1.1$, we have:

$$y = 1.1e^{kt}$$

$$3 = 1.1e^{(k)(-3)}$$

$$\ln 3 = \ln 1.1 - 3k$$

$$k = \frac{1}{3}(\ln 1.1 - \ln 3)$$

Function: $y = 1.1e^{(\ln 1.1 - \ln 3)t/3}$ or

$y \approx 1.1e^{-0.3344t}$

29. At time $t = \frac{3}{k}$, the amount remaining is $y_0 e^{-kt} = y_0 e^{-k(3/k)} = y_0 e^{-3} \approx 0.0498y_0$. This is less than 5% of the original amount, which means that over 95% has decayed already.

30.

$$T - T_s = (T_0 - T_s)\,e^{-kt}$$

$$35 - 65 = (T_0 - 65)e^{-(k)(10)}$$

$$50 - 65 = (T_0 - 65)e^{-(k)(20)}$$

Dividing the first equation by the second, we have:

$$2 = e^{10k}$$

$$k = \frac{1}{10}\ln 2$$

Substituting back into the first equation, we have:

$$-30 = (T_0 - 65)e^{-[(\ln 2)/10](10)}$$

$$-30 = (T_0 - 65)\left(\frac{1}{2}\right)$$

$$-60 = T_0 - 65$$

$$5 = T_0$$

The beam's initial temperature is 5°F.

31. **(a)** First, we find the value of k.

$$T - T_s = (T_0 - T_s)\,e^{-kt}$$
$$60 - 20 = (90 - 20)\,e^{-(k)(10)}$$
$$\frac{4}{7} = e^{-10k}$$
$$k = -\frac{1}{10}\ln\frac{4}{7}$$

When the soup cools to 35°, we have:

$$35 - 20 = (90 - 20)\,e^{[(1/10)\ln(4/7)]t}$$
$$15 = 70e^{[(1/10)\ln(4/7)]t}$$
$$\ln\frac{3}{14} = \left(\frac{1}{10}\ln\frac{4}{7}\right)t$$
$$t = \frac{10\ln\left(\frac{3}{14}\right)}{\ln\left(\frac{4}{7}\right)} \approx 27.53 \text{ min}$$

It takes a total of about 27.53 minutes, which is an additional 17.53 minutes after the first 10 minutes.

(b) Using the same value of k as in part (a), we have:

$$T - T_s = (T_0 - T_s)e^{-kt}$$
$$35 - (-15) = [90 - (-15)]\,e^{[(1/10)\ln(4/7)]t}$$
$$50 = 105e^{[(1/10)\ln(4/7)]t}$$
$$\ln\frac{10}{21} = \left(\frac{1}{10}\ln\frac{4}{7}\right)t$$
$$t = \frac{10\ln\left(\frac{10}{21}\right)}{\ln\left(\frac{4}{7}\right)} \approx 13.26$$

It takes about 13.26 minute

32. First, we find the value of k. Taking "right now" as $t = 0$, 60° above room temperature means $T_0 - T_s = 60$. Thus, we have

$$T - T_s = (T_0 - T_s)\,e^{-kt}$$
$$70 = 60e^{(-k)(-20)}$$
$$\frac{7}{6} = e^{20k}$$
$$k = \frac{1}{20}\ln\frac{7}{6}$$

(a) $T - T_s = (T_0 - T_s)\ e^{-kt}$

$$= 60e^{(-(1/20)\ln(7/6))(15)} \approx 53.45$$

It will be about 53.45°C above room temperature.

(b) $T - T_s = (T_0 - T_s)\ e^{-kt}$

$$= 60e^{(-(1/20)\ln(7/6))(120)} \approx 23.79$$

It will be about 23.79° above room temperature.

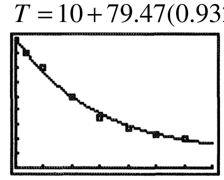
(c) $T - T_s = (T_0 - T_s)\,e^{-kt}$

$$10 = 60e^{(-(1/20)\ln(7/6))t}$$
$$\ln\frac{1}{6} = \left(-\frac{1}{20}\ln\frac{7}{6}\right)t$$
$$t = -\frac{20\ln\left(\frac{1}{6}\right)}{\ln\left(\frac{7}{6}\right)} \approx 232.47 \text{ min}$$

It will take about 232.47 min or 3.9 hr.

33. **(a)** $T - T_s = 79.47(0.932)^t$

(b) $T = 10 + 79.47(0.932)^t$



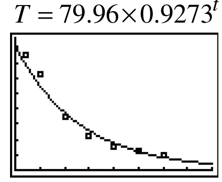
[0, 35] by [0, 90]

(c) Solving $T = 12$ and using the exact values from the regression equation, we obtain $t \approx 52.5$ sec.

(d) Substituting $t = 0$ into the equation we found in part (b), the temperature was approximately 89.47°C.

34. **(a)** Newton's Law of Cooling predicts that the difference between the probe temperature (T) and the surrounding temperature (T_s) is an exponential function of time, but in this case $T_s = 0$, so T is an exponential function of time.

(b) $T = 79.96 \times 0.9273^t$



[–0, 40] by [0, 86]

(c) At about 37 seconds.

(d) 76.96°C

35. Use $k = \frac{\ln 2}{5700}$ (see Example 5).

$$e^{-kt} = 0.445$$
$$-kt = \ln 0.445$$
$$t = -\frac{\ln 0.445}{k}$$
$$= -\frac{5700 \ln 0.445}{\ln 2} \approx 6658 \text{ years}$$

Crater Lake is about 6658 years old.

36. Use $k = \frac{\ln 2}{5700}$ (see Example 5).

(a) $e^{-kt} = 0.17$

$$-kt = \ln 0.17$$
$$t = -\frac{\ln 0.17}{k}$$
$$= -\frac{5700 \ln 0.17}{\ln 2} \approx 14{,}571 \text{ years}$$

The animal died about 14,571 years before A.D. 2000, in 12,571 B.C.E.

(b) $e^{-kt} = 0.18$

$$-kt = \ln 0.18$$
$$t = -\frac{\ln 0.18}{k}$$
$$= -\frac{5700 \ln 0.18}{\ln 2} \approx 14{,}101 \text{ years}$$

The animal died about 14,101 years before A.D. 2000, in 12,101 B.C.E.

(c) $e^{-kt} = 0.16$

$$-kt = \ln 0.16$$
$$t = -\frac{\ln 0.16}{k}$$
$$= -\frac{5700 \ln 0.16}{\ln 2} \approx 15{,}070 \text{ years}$$

The animal died about 15,070 years before A.D. 2000, in 13,070 B.C.E.

37.
$$\frac{1}{3} = e^{-kt}$$
$$k = \frac{\ln\left(\frac{1}{3}\right)}{5} = 0.22$$
$$-\frac{\ln\left(\frac{1}{3}\right)}{5} = -0.22$$
$$\frac{1}{2} = e^{-0.22t}$$
$$t = \frac{\ln\left(\frac{1}{2}\right)}{-0.22} = 3.15 \text{ years}$$

38. $3 = e^{rt}$

$$r = \frac{\ln(3)}{10} = 0.11$$
$$4 = e^{0.11t}$$
$$t = \frac{\ln(4)}{0.11} = 12.60 \text{ years}$$

39. $y = y_0 e^{-kt}$

$$800 = 1000e^{-(k)(10)}$$
$$0.8 = e^{-10k}$$
$$k = -\frac{\ln 0.8}{10}$$

At $t = 10 + 14 = 24$ h:

$$y = 1000e^{-(-\ln 0.8/10)24}$$
$$= 1000e^{2.4 \ln 0.8} \approx 585.4 \text{ kg}$$

About 585.4 kg will remain.

40. $0.2 = e^{-0.1t}$

$$\ln 0.2 = -0.1t$$
$$t = -10 \ln 0.2 \approx 16.09 \text{ yr}$$

It will take about 16.09 years.

41. **(a)**
$$\frac{dp}{dh} = kp$$
$$\frac{dp}{p} = k\ dh$$
$$\int \frac{dp}{p} = \int k\ dh$$
$$\ln|p| = kh + C$$
$$e^{\ln|p|} = e^{kh+C}$$
$$|p| = e^C e^{kh}$$
$$p = Ae^{kh}$$

Initial condition: $p = p_0$ when $h = 0$

$$p_0 = Ae^0$$
$$A = p_0$$

Solution: $p = p_0 e^{kh}$

Using the given altitude-pressure data, we have $p_0 = 1013$ millibars, so:

$$p = 1013e^{kh}$$
$$90 = 1013e^{(k)(20)}$$
$$\frac{90}{1013} = e^{20k}$$
$$k = \frac{1}{20} \ln \frac{90}{1013} \approx -0.121 \text{ km}^{-1}$$

Thus, we have $p \approx 1013e^{-0.121h}$.

(b) At 50 km, the pressure is
$1013e^{((1/20)\ln(90/1013))(50)}$
≈ 2.383 millibars.

(c)
$$900 = 1013e^{kh}$$
$$\frac{900}{1013} = e^{kh}$$
$$h = \frac{1}{k}\ln\frac{900}{1013}$$
$$= \frac{20\ln\left(\frac{900}{1013}\right)}{\ln\left(\frac{90}{1013}\right)} \approx 0.977 \text{ km}$$
The pressure is 900 millibars at an altitude of about 0.977 km.

42. By the Law of Exponential Change, $y = 100e^{-0.6t}$. At $t = 1$ hour, the amount remaining will be $100e^{-0.6(1)} \approx 54.88$ grams.

43. (a) By the Law of Exponential Change, the solution is $V = V_0e^{-(1/40)t}$.

(b)
$$0.1 = e^{-(1/40)t}$$
$$\ln 0.1 = -\frac{t}{40}$$
$$t = -40\ln 0.1 \approx 92.1 \text{ sec}$$
It will take about 92.1 seconds.

44. (a) $A(t) = A_0e^t$
It grows by a factor of e each year.

(b)
$$3 = e^t$$
$$\ln 3 = t$$
It will take $\ln 3 \approx 1.1$ yr.

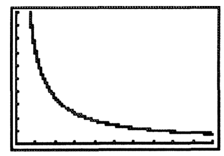
(c) In one year your account grows from A_0 to A_0e, so you can earn $A_0e - A_0$, or $(e - 1)$ times your initial amount. This represents an increase of about 172%.

45. (a)
$$90 = e^{(r)(100)}$$
$$\ln 90 = 100r$$
$$r = \frac{\ln 90}{100} \approx 0.045 \text{ or } 4.5\%$$

(b)
$$131 = e^{(r)(100)}$$
$$\ln 131 = 100r$$
$$r = \frac{\ln 131}{100} \approx 0.049 \text{ or } 4.9\%$$

46. (a)
$$2y_0 = y_0e^{rt}$$
$$2 = e^{rt}$$
$$\ln 2 = rt$$
$$t = \frac{\ln 2}{r}$$

(b)



[0, 0.1] by [0, 100]

(c) $\ln 2 \approx 0.69$, so the doubling time is $\frac{0.69}{r}$ which is almost the same as the rules.

(d) $\frac{70}{5} = 14$ years or $\frac{72}{5} = 14.4$ years

(e)
$$3y_0 = y_0e^{rt}$$
$$3 = e^{rt}$$
$$\ln 3 = rt$$
$$t = \frac{\ln 3}{r}$$
Since ln 3 ≈ 1.099, a suitable rule is $\frac{108}{100r}$ or $\frac{108}{i}$.
(We choose 108 instead of 110 because 108 has more factors.)

47. False; the correct solution is $|y| = e^{kx+C}$, which can be written (with a new C) as $y = Ce^{kx}$.

48. True; the differential equation is solved by an exponential equation that can be written in any base. Note that $Ce^{2t} = C(3^{kt})$ when $k = \frac{2}{\ln 3}$.

49. D; $A(t) = A_0e^{rt}$
$$2 = 1e^{7r}$$
$$r = \frac{\ln(2)}{7} = 0.099$$
$$t = \frac{\ln(3)}{0.099} = 11.1$$

50. C;
$$A = A_0\left(\frac{1}{2}\right)^{t/r}$$
$$1 = 100\left(\frac{1}{2}\right)^{199/r}$$
$$\ln(.01) = \frac{199}{r}\ln(0.5)$$
$$r = \frac{199\ln(0.5)}{\ln(0.01)} = 30$$

51. D

52. E; $T - 68 = (425 - 68)e^{-kt}$
$$195 - 68 = 357e^{-30k}$$
$$e^{-30k} = \frac{127}{357} = 0.356$$
$$k = \frac{\ln(0.356)}{-30} = .0344$$
$$100 = 68 + 357e^{(-0.0344)t}$$
$$t = \frac{\ln\left(\frac{100-68}{357}\right)}{-0.0344} = 70 \text{ min}$$
$70 - 30 = 40$

53. **(a)** Since acceleration is $\frac{dv}{dt}$, we have

$$\text{Force} = m\frac{dv}{dt} = -kv.$$

(b) From $m\frac{dv}{dt} = -kv$ we get $\frac{dv}{dt} = -\frac{k}{m}v$, which is the differential equation for exponential growth modeled by $v = Ce^{-(k/m)t}$. Since $v = v_0$ at $t = 0$, it follows that $C = v_0$.

(c) In each case, we would solve $2 = e^{-(k/m)t}$. If k is constant, an increase in m would require an increase in t. The object of larger mass takes longer to slow down. Alternatively, one can consider the equation $\frac{dv}{dt} = -\frac{k}{m}v$ to see that v changes more slowly for larger values of m.

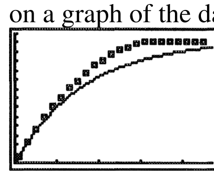
54. **(a)** $s(t) = \int v_0 e^{-(k/m)t}\,dt = -\frac{v_0 m}{k}e^{-(k/m)t} + C$

Initial condition: $s(0) = 0$
$$0 = -\frac{v_0 m}{k} + C$$
$$\frac{v_0 m}{k} = C$$
$$s(t) = -\frac{v_0 m}{k}e^{-(k/m)t} + \frac{v_0 m}{k}$$
$$= \frac{v_0 m}{k}(1 - e^{-(k/m)t})$$

(b) $\lim_{t\to\infty} s(t) = \lim_{t\to\infty} \frac{v_0 m}{k}(1 - e^{-(k/m)t}) = \frac{v_0 m}{k}$

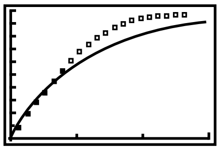
55.
$$\frac{v_0 m}{k} = \text{coasting distance}$$
$$\frac{(0.80)(49.90)}{k} = 1.32$$
$$k = \frac{998}{33}$$
We know that $\frac{v_0 m}{k} = 1.32$ and $\frac{k}{m} = \frac{998}{33(49.9)} = \frac{20}{33}$.

We have:
$$s(t) = \frac{v_0 m}{k}(1 - e^{-(k/m)t})$$
$$= 1.32(1 - e^{-20t/33})$$
$$\approx 1.32(1 - e^{-0.606t})$$
A graph of the model is shown superimposed on a graph of the data.



[0, 4.7] by [0, 1.4]

56.
$$\frac{v_0 m}{k} = \text{coasting distance}$$
$$\frac{(0.86)(30.84)}{k} = 0.97$$
$$k \approx 27.343$$
$$s(t) = \frac{v_0 m}{k}(1 - e^{-(k/m)t})$$
$$s(t) = 0.97(1 - e^{-(27.343/30.84)t})$$
$$s(t) = 0.97(1 - e^{-0.8866t})$$
A graph of the model is shown superimposed on a graph of the data.



[0, 3] by [0, 1]

57. (a)

x	$\left(1+\frac{1}{x}\right)^x$
10	2.5937
100	2.7048
1000	2.7169
10,000	2.7181
100,000	2.7183

$e \approx 2.7183$

(b) $r = 2$

x	$\left(1+\frac{2}{x}\right)^x$
10	6.1917
100	7.2446
1000	7.3743
10,000	7.3876
100,000	7.3889

$e^2 \approx 7.389$

$r = 0.5$

x	$\left(1+\frac{0.5}{x}\right)^x$
10	1.6289
100	1.6467
1000	1.6485
10,000	1.6487
100,000	1.6487

$e^{0.5} \approx 1.6487$

(c) As we compound more times, the increment of time between compounding approaches 0. Continuous compounding is based on an instantaneous rate of change which is a limit of average rates as the increment in time approaches 0.

58. (a) To simplify calculations somewhat, we may write:

$$v(t) = \sqrt{\frac{mg}{k}}\frac{(e^{at}-e^{-at})e^{at}}{(e^{at}+e^{-at})e^{at}}$$
$$= \sqrt{\frac{mg}{k}}\frac{e^{2at}-1}{e^{2at}+1}$$
$$= \sqrt{\frac{mg}{k}}\frac{(e^{2at}+1)-2}{e^{2at}+1}$$
$$= \sqrt{\frac{mg}{k}}\left(1-\frac{2}{e^{2at}+1}\right)$$

The left side of the differential equation is:

$$m\frac{dv}{dt} = m\sqrt{\frac{mg}{k}}(2)(e^{2at}+1)^{-2}(2ae^{2at})$$
$$= 4ma\sqrt{\frac{mg}{k}}(e^{2at}+1)^{-2}(e^{2at})$$
$$= 4m\sqrt{\frac{gk}{m}}\sqrt{\frac{mg}{k}}(e^{2at}+1)^{-2}(e^{2at})$$
$$= \frac{4mge^{2at}}{(e^{2at}+1)^2}$$

The right side of the differential equation is:

$$mg - kv^2$$
$$= mg - k\left(\frac{mg}{k}\right)\left(1-\frac{2}{e^{2at}+1}\right)^2$$
$$= mg\left[1-\left(1-\frac{2}{e^{2at}+1}\right)^2\right]$$
$$= mg\left(1-1+\frac{4}{e^{2at}+1}-\frac{4}{(e^{2at}+1)^2}\right)$$
$$= mg\frac{4(e^{2at}+1)-4}{(e^{2at}+1)^2}$$
$$= \frac{4\,mg\,e^{2at}}{(e^{2at}+1)^2}$$

Since the left and right sides are equal, the differential equation is satisfied.

And $v(0) = \sqrt{\frac{mg}{k}}\frac{e^0-e^0}{e^0+e^0} = 0,$ so the initial condition is also satisfied.

(b) $\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\left(\sqrt{\frac{mg}{k}}\frac{e^{at}-e^{-at}}{e^{at}+e^{-at}} \bullet \frac{e^{-at}}{e^{-at}}\right)$

$$= \lim_{t\to\infty}\left(\sqrt{\frac{mg}{k}}\frac{1-e^{-2at}}{1+e^{-2at}}\right)$$
$$= \sqrt{\frac{mg}{k}}\left(\frac{1-0}{1+0}\right)$$
$$= \sqrt{\frac{mg}{k}}$$

The limiting velocity is $\sqrt{\frac{mg}{k}}$.

(c) $\sqrt{\frac{mg}{k}} = \sqrt{\frac{160}{0.005}} \approx 179 \text{ ft/sec}$

The limiting velocity is about 179 ft/sec, or about 122 mi/hr.

Section 7.5 Logistic Growth (pp. 366–375)

Exploration 1 Exponential Growth Revisited

1. $100(2)^{12} = 409,600$

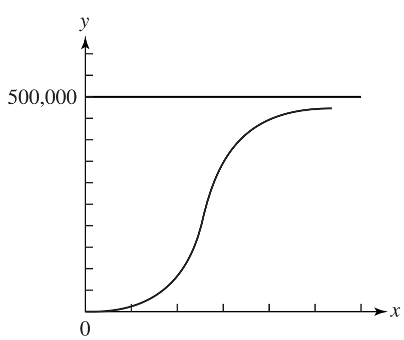
2. $100(2)^{(12\cdot 24)} = 4.97\times 10^{88}$

3. No; this number is much larger than the estimated number of atoms.

4. $500,000 = 100(2)^x$

   $\frac{\log 5000}{\log 2} = x = 12.29$ hours

5.



Exploration 2 Learning From the Differential Equation

1. $\frac{dP}{dt}$ will be close to zero when P is close to 0 and when P is close to M.

2. P is half the value of M at its vertex.

3. When $P = \frac{M}{2}$, $\frac{dP}{dt} = kP(M-P)$ is at its maximum.

4. When the initial population is less than M, the initial growth rate is positive.

5. When the initial population is more than M, the initial growth rate is negative.

6. When the initial population is equal to M, the growth rate is 0.

7. $\lim_{t\to\infty} P(t) = M$, regardless of the initial population. The limit depends only on M.

Quick Review 7.5

1. $$\begin{array}{r|l} & x+1 \\ \hline x-1 & x^2 \\ & \underline{x^2-x} \\ & x \\ & \underline{x-1} \\ & 1 \end{array}$$

   $x+1+\frac{1}{x-1}$

2. $$\begin{array}{r|l} & 1 \\ \hline x^2-4 & x^2 \\ & \underline{x^2-4} \\ & 4 \end{array}$$

   $1+\frac{4}{x^2-4}$

3. $$\begin{array}{r|l} & 1 \\ \hline x^2+x-2 & x^2+x+1 \\ & \underline{x^2+x-2} \\ & 3 \end{array}$$

   $1+\frac{3}{x^2+x-2}$

4. $$\begin{array}{r|l} & x \\ \hline x^2-1 & x^3 \quad -5 \\ & \underline{x^3-x} \\ & x-5 \end{array}$$

   $x+\frac{x-5}{x^2-1}$

5. $(-\infty, \infty)$

6. $\lim\limits_{x\to\infty} \dfrac{60}{1+5e^{-0.1x}} = \dfrac{60}{1+5(0)} = 60$

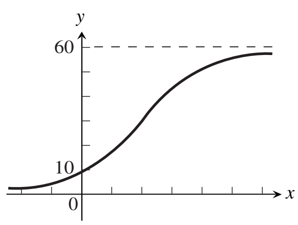
7. As $x \to -\infty$, $-0.1x \to \infty$, and $e^{-0.1x} \to \infty$, so

$\lim\limits_{x\to-\infty} \dfrac{60}{1+5e^{-0.1x}} = 0.$

8. $y(0) = \dfrac{60}{1+5\ e^{(-0.1(0))}} = 10$

9. From problems 6 and 7, the two horizontal asymptotes are $y = 0$ and $y = 60$.

10.



Section 7.5 Exercises

1. $A(x-4)+B(x) = x-12$

$$
\begin{aligned}
x = 4,\quad 4B &= -8\\
B &= -2\\
x = 1,\ A(1-4)+(-2)(1) &= 1-12\\
A &= 3
\end{aligned}
$$

2. $A(x-2)+B(x+3) = 2x+16$

$$
\begin{aligned}
x = 2,\ B(2+3) &= 2(2)+16\\
5B &= 20\\
B &= 4\\
x = -3,\ A(-3-2) &= 2(-3)+16\\
-5A &= 10\\
A &= -2
\end{aligned}
$$

3. $A(x+5)+B(x-2) = 16-x$

$$
\begin{aligned}
x = -5,\ B(-5-2) &= 16-(-5)\\
-7B &= 21\\
B &= -3\\
x = 2,\ A(2+5) &= 16-2\\
7A &= 14\\
A &= 2
\end{aligned}
$$

4. $A(x+3)+B(x-3) = 3$

$$
\begin{aligned}
x = -3,\ B(-3-3) &= 3\\
-6B &= 3\\
B &= -\frac{1}{2}\\
x = 3,\ A(3+3) &= 3\\
6A &= 3\\
A &= \frac{1}{2}
\end{aligned}
$$

5. See problem 1.

$$
\begin{aligned}
\int \frac{x-12}{x^2-4}\,dx &= \int\left(\frac{3}{x}+\frac{-2}{x-4}\right)dx\\
&= 3\ \ln|x| - 2\ \ln|x-4| + C\\
&= \ln\left(\frac{|x|^3}{(x-4)^2}\right) + C
\end{aligned}
$$

6. See problem 2.

$$
\begin{aligned}
&\int \frac{2x+16}{x^2+x-6}\,dx\\
&= \int\left(\frac{-2}{x+3}+\frac{4}{x-2}\right)dx\\
&= -2\ln|x+3| + 4\ln|x-2| + C\\
&= \ln\left(\frac{(x-2)^4}{(x+3)^2}\right) + C
\end{aligned}
$$

7. $x^2-4\overline{\smash{)}2x^3}$ with quotient $2x$

$$
\begin{array}{r}
2x^3-8x\\
\hline
8x
\end{array}
$$

$$\int\left(2x+\frac{8x}{x^2-4}\right)dx$$

$u = x^2-4$

$du = 2x\ dx$

$$
\begin{aligned}
x^2+4\int\frac{du}{u} &= x^2+4\ln|u| + C\\
&= x^2+\ln(x^2-4)^4 + C
\end{aligned}
$$

8. $x^2-9\overline{\smash{)}x^2-6}$ with quotient 1

$$
\begin{array}{r}
x^2-9\\
\hline
3
\end{array}
$$

$$\int 1+\frac{3}{x^2-9}\,dx = x+\int\frac{A}{x+3}+\frac{B}{x-3}\,dx$$

$A(x-3)+B(x+3) = 3$

$$x = 3,\ B(3+3) = 3$$
$$B = \frac{1}{2}$$
$$x = -3,\ A(-3-3) = 3$$
$$A = -\frac{1}{2}$$
$$x + \int \frac{-\frac{1}{2}}{x+3} + \frac{\frac{1}{2}}{x-3}\,dx = x + \ln\sqrt{\left|\frac{x-3}{x+3}\right|} + C$$

9. $2\int \frac{dx}{x^2+1} = 2\tan^{-1} x + C$

10. $3\int \frac{dx}{x^2+9} = \tan^{-1}\left(\frac{x}{3}\right) + C$

11. $\int \frac{7}{2x^2-5x-3}\,dx$

$$\frac{A}{2x+1} + \frac{B}{x-3} = \frac{7}{(2x+1)(x-3)}$$
$$A(x-3) + B(2x+1) = 7$$
$$x = 3,\ B(2(3)+1) = 7$$
$$B = 1$$
$$x = -\frac{1}{2},\ A\left(-\frac{1}{2}-3\right) = 7$$
$$A = -2$$
$$\int\left(\frac{-2}{2x+1} + \frac{1}{x-3}\right)dx = \ln\left|\frac{x-3}{2x+1}\right| + C$$

12. $\int \frac{1-3x}{3x^2-5x-3}\,dx$

$$\frac{A}{3x-2} + \frac{B}{x-1} = \frac{1-3x}{(3x-2)(x-1)}$$
$$A(x-1) + B(3x-2) = 1-3x$$
$$x = 1,\ B(3(1)-2) = 1-3(1)$$
$$B = -2$$
$$x = \frac{2}{3},$$
$$A = \left(\frac{2}{3}-1\right) = 1-3\left(\frac{2}{3}\right)$$
$$-\frac{1}{3}A = -1$$
$$A = 3$$
$$\int\left(\frac{3}{3x-2} + \frac{-2}{x-1}\right)dx$$
$$= \ln|3x-2| - 2\ln|x-1| + C$$
$$= \ln\left|\frac{3x-2}{(x-1)^2}\right| + C$$

13. $\int \frac{8x-7}{2x^2-x-3}$

$$\frac{A}{x+1} + \frac{B}{2x-3} = \frac{8x-7}{(x+1)(2x-3)}$$
$$A(2x-3) + B(x+1) = 8x-7$$
$$x = \frac{3}{2},\ B\left(\frac{3}{2}+1\right) = 8\left(\frac{3}{2}\right) - 7$$
$$\frac{5}{2}B = 5$$
$$B = 2$$
$$x = -1,\ A(2x-3) = 8x-7$$
$$A(-2-3) = 8-7$$
$$-5A = -15$$
$$A = 3$$
$$\int\left(\frac{3}{x+1} + \frac{2}{2x-3}\right)dx$$
$$= 3\ln|x+1| + \ln|2x-3| + C$$
$$= \ln\left(|x+1|^3\,|2x-3|\right) + C$$

14. $\int \frac{5x+14}{x^2+7x}\,dx$

$$\frac{A}{x} + \frac{B}{x+7} = \frac{5x+14}{x(x+7)}$$
$$A(x+7) + Bx = 5x+14$$
$$x = -7,\ -7B = 5(-7)+14$$
$$-7B = -21$$
$$B = 3$$
$$x = 0,\ A(0+7) = 5(0)+14$$
$$7A = 14$$
$$A = 2$$
$$\int\left(\frac{2}{x} + \frac{3}{x+7}\right)dx = 2\ln|x| + 3\ln|x+7| + C$$
$$= \ln\left(x^2|x+7|^3\right) + C$$

15. $\int dy = \int \frac{2x-6}{x^2-2x}\,dx$

$$\frac{A}{x} + \frac{B}{x-2} = \frac{2x-6}{x(x-2)}$$
$$A(x-2) + Bx = 2x-6$$
$$x = 2,\ 2B = 2(2)-6$$
$$2B = -2$$
$$B = -1$$

$$x = 0,\ A(0-2) = 2(0) - 6$$
$$-2A = -6$$
$$A = 3$$
$$\int \left(\frac{3}{x} + \frac{-1}{x-2} \right) dx$$
$$y = 3 \ln|x| - \ln|x-2| + C$$
$$y = \ln\left| \frac{x^3}{x-2} \right| + C$$

16. $\int du = \int \frac{2}{x^2-1}\, dx$

$$\frac{A}{x+1} + \frac{B}{x-1} = \frac{2}{(x+1)(x-1)}$$
$$A(x-1) + B(x+1) = 2$$
$$x = 1,\ B(1+1) = 2$$
$$2B = 2$$
$$B = 1$$
$$x = -1,\ A(-1-1) = 2$$
$$-2A = 2$$
$$A = -1$$
$$u = \int \left(\frac{-1}{x+1} + \frac{1}{x-1} \right) dx$$
$$u = -\ln|x+1| + \ln|x-1| + C$$
$$u = \ln\left| \frac{x-1}{x+1} \right| + C$$

17. $\int F'(x)\, dx = \int \frac{2}{x^3 - x}\, dx$

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} = \frac{2}{x(x+1)(x-1)}$$
$$A(x+1)(x-1) + Bx(x-1) + Cx(x+1) = 2$$
$$x = 1,\ 2C = 2$$
$$C = 1$$
$$x = -1,\ 2B = 2$$
$$B = 1$$
$$x = 0,\ -A = 2$$
$$A = -2$$
$$\int \left(\frac{-2}{x} + \frac{1}{x+1} + \frac{1}{x-1} \right) dx$$
$$F(x) = -2\ln|x| + \ln|x+1| + \ln|x-1| + C$$
$$F(x) = \ln\left(\frac{|x^2-1|}{x^2} \right) + C$$

18. $\int G'(t)\, dt = \int \frac{2t^3}{t^3 - t}\, dt$

$$\begin{array}{r} 2 \\ t^3 - t \overline{)\,2t^3} \\ \underline{2t^3 - 2t} \\ 2t \end{array}$$

$$= \int 2 + \frac{2t}{t^3 - t}\, dt$$
$$= 2t + \int \frac{2}{t^2 - 1}\, dt$$
$$\frac{A}{t-1} + \frac{B}{t+1} = \frac{2}{(t-1)(t+1)}$$
$$A(t+1) + B(t-1) = 2$$
$$t = -1,\ B(-1-1) = 2$$
$$-2B = 2$$
$$B = -1$$
$$t = 1,\ A(1+1) = 2$$
$$2A = 2$$
$$A = 1$$
$$G(t) = 2t + \int \left(\frac{1}{(t-1)} + \frac{-1}{(t+1)} \right) dt$$
$$= 2t + \ln|t-1| - \ln|t+1| + C$$
$$= 2t + \ln\left| \frac{t-1}{t-1} \right| + C$$

19. $\int \frac{2x}{x^2-4}\, dx$

$$u = x^2 - 4$$
$$du = 2x\, dx$$
$$\int \frac{du}{u} = \ln u + C = \ln|x^2 - 4| + C$$

20. $\int \frac{4x-3}{2x^2 - 3x + 1}\, dx$

$$u = 2x^2 - 3x + 1$$
$$du = (4x-3)\, dx$$
$$\int \frac{du}{u} = \ln u + C = \ln|2x^2 - 3x + 1| + C$$

21. $\int \frac{x^2+x-1}{x^2-x}\,dx$

$$\begin{array}{r} 1 \\ x^2-x\overline{)x^2+x-1} \\ \underline{x^2-x} \\ 2x-1 \end{array}$$

$$\int\left(1+\frac{2x-1}{x^2-x}\right)dx$$

$u = x^2 - x$

$du = (2x-1)\,dx$

$$x+\int\frac{du}{u} = x\ln u + C = x + \ln\left|x^2-x\right| + C$$

22. $\int \frac{2x^3}{x^2-1}\,dx$

$$\begin{array}{r} 2x \\ x^2-1\overline{)2x^3} \\ \underline{2x^3-2x} \\ 2x \end{array}$$

$$\int\left(2x+\frac{2x}{x^2-1}\right)dx$$

$u = x^2 - 1$

$du = 2x\,dx$

$$x^2+\int\frac{du}{u} = x^2 + \ln u + C = x^2 + \ln\left|x^2-1\right| + C$$

23. **(a)** 200 individuals

(b) 100 individuals

(c) $\frac{dP(100)}{dt} = 0.006(100)(200-100)$
$= 60$ individuals per year.

24. **(a)** 700 individuals

(b) 350 individuals

(c) $\frac{dP(350)}{dt} = 0.0008(350)(700-350)$
$= 98$ individuals per year.

25. **(a)** 1200 individuals

(b) 600 individuals

(c) $\frac{dP(600)}{dt} = 0.0002(600)(1200-600)$
$= 72$ individuals per year.

26. **(a)** 5000 individuals

(b) 2500 individuals

(c) $\frac{dP(2500)}{dt} = 10^{-5}(2500)(5000-2500)$
$= 62.5$ individuals per year.

27. $\frac{dP}{dt} = 0.006\ \ P(200-P)$

$$\int\frac{dP}{P(200-P)} = \int 0.006\,dt$$

$$\frac{A}{P}+\frac{B}{200-P} = \frac{1}{P(200-P)}$$

$A(200-P)+BP = 1$

$P = 200,\ 200B = 1$

$B = 0.005$

$P = 0,\ A(200-0) = 1$

$200A = 1$

$A = 0.005$

$$\int\left(\frac{0.005}{P}+\frac{0.005}{200-P}\right)dP = 0.006t$$

$$\int\left(\frac{1}{P}+\frac{1}{200-P}\right)dP = 1.2t$$

$\ln P - \ln\,(200-P) = 1.2t + C$

$$\ln\left(\frac{200-P}{P}\right) = -1.2t - C$$

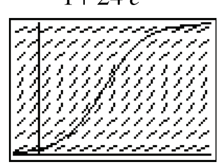
$$\frac{200}{P} - 1 = e^{-1.2t}\ e^{-c}$$

$$\frac{200}{P} = 1 + e^{-1.2t}\ e^{-c}$$

$$\frac{200}{8} = 1 + e^{-1.2(0)}\ e^{-c}$$

$e^{-c} = 24$

$$P = \frac{200}{1+24\,e^{-1.2t}}$$



[–1, 7] by [0, 200]

28. $\dfrac{dP}{dt} = 0.0008 \;\; P(700 - P)$

$$\int \frac{dP}{P(700-P)} = \int 0.0008\, dt$$

$$\frac{A}{P} + \frac{B}{700-P} = \frac{1}{P(700-P)}$$

$$A(700-P) + BP = 1$$

$$P = 700,\;\; 700B = 1$$

$$B = \frac{1}{700}$$

$$P = 0,\;\; A(700-0) = 1$$

$$A = \frac{1}{700}$$

$$\int \left(\frac{\frac{1}{700}}{P} + \frac{\frac{1}{700}}{700-P} \right) dP = 0.0008t$$

$$\int \left(\frac{1}{P} + \frac{1}{700-P} \right) dP = 0.56t$$

$$\ln P - \ln(700-P) = 0.56t + C$$

$$\ln\left(\frac{700-P}{P} \right) = -0.56t - C$$

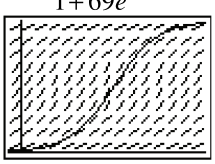
$$\frac{700}{P} - 1 = e^{-0.56t} \;\; e^{-c}$$

$$\frac{700}{P} = 1 + e^{-0.56t} \;\; e^{-c}$$

$$\frac{700}{10} = 1 + e^{-0.56(0)} \;\; e^{-c}$$

$$e^{-c} = 69$$

$$P = \frac{700}{1 + 69e^{-0.56t}}$$



[–1, 15] by [0, 700]

29. $\dfrac{dP}{dt} = 0.0002P(1200 - P)$

$$\int \frac{dP}{P(1200-P)} = \int 0.0002\, dt$$

$$\frac{A}{P} + \frac{B}{1200-P} = \frac{1}{P(1200-P)}$$

$$A(1200-P) + BP = 1$$

$$P = 1200,\;\; 1200B = 1$$

$$B = \frac{1}{1200}$$

$$P = 0,\;\; A(1200-0) = 1$$

$$1200A = 1$$

$$A = \frac{1}{1200}$$

$$\int \left(\frac{\frac{1}{1200}}{P} + \frac{\frac{1}{1200}}{700-P} \right) dP = 0.0002t$$

$$\int \left(\frac{1}{P} + \frac{1}{1200-P} \right) dP = 0.24t$$

$$\ln P - \ln(1200-P) = 0.24t + C$$

$$\ln\left(\frac{1200-P}{P} \right) = -0.24\,t - C$$

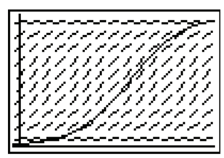
$$\frac{1200}{P} - 1 = e^{-0.24t} e^{-c}$$

$$\frac{1200}{P} = 1 + e^{-0.24t} e^{-c}$$

$$\frac{1200}{20} = 1 + e^{-0.24(0)} e^{-c}$$

$$e^{-c} = 59$$

$$P = \frac{1200}{1 + 59\, e^{-0.24t}}$$



[–1, 30] by [0, 1200]

30. $\dfrac{dP}{dt} = 10^{-5} \;\; P(5000 - P)$

$$\int \frac{dP}{P(5000-P)} = \int 10^{-5}\; dt$$

$$\frac{A}{P} + \frac{B}{5000-P} = \frac{1}{P(5000-P)}$$

$$A(5000-P) + BP = 1$$

$$P = 5000,\;\; 5000B = 1$$

$$B = 0.0002$$

$$P = 0,\;\; A(5000-0) = 1$$

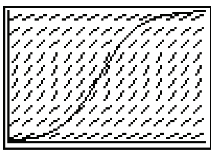
$$5000A = 1$$

$$A = 0.0002$$

$$\int \left(\frac{0.0002}{P} + \frac{0.0002}{5000-P} \right) dP = 10^{-5} t$$

$$\int \left(\frac{1}{P} + \frac{1}{5000-P} \right) dP = 0.05t$$

$$\ln P - \ln(5000 - P) = 0.05t + C$$
$$\ln\left(\frac{5000 - P}{P}\right) = -0.05t - C$$
$$\frac{5000}{P} - 1 = e^{-0.05t}e^{-c}$$
$$\frac{5000}{P} = 1 + e^{-0.05t}e^{-c}$$
$$\frac{5000}{50} = 1 + e^{-0.05(0)}e^{-c}$$
$$e^{-c} = 99$$
$$P = \frac{5000}{1 + 99e^{-0.05t}}$$



[–1, 200] by [0, 5000]

31. **(a)** $P(t) = \dfrac{1000}{1 + e^{4.8 - 0.7t}}$
$= \dfrac{1000}{1 + e^{4.8 - 0.7t}}$
$= \dfrac{M}{1 + Ae^{-Mkt}}$

This is a logistic growth model with $M = 1000$ and $k = \dfrac{0.7}{1000} = 0.0007.$

(b) $P(0) = \dfrac{1000}{1 + e^{4.8}} \approx 8$

Initially there are 8 rabbits.

32. **(a)** $P(t) = \dfrac{200}{1 + e^{5.3 - t}}$
$= \dfrac{200}{1 + e^{5.3}e^{-t}}$
$= \dfrac{M}{1 + Ae^{-Mkt}}$

This is a logistic growth model with $M = 200$ and $k = \dfrac{1}{200} = 0.05.$

(b) $P(0) = \dfrac{200}{1 + e^{5.3}} \approx 1$

Initially 1 student has the measles.

33. **(a)** $\dfrac{dP}{dt} = 0.0015P(150 - P)$
$= kP(M - P)$

Thus, $k = 0.0015$ and $M = 150$.

$P = \dfrac{M}{1 + Ae^{-Mkt}} = \dfrac{150}{1 + Ae^{-0.225t}}$

Initial condition: $P(0) = 6$

$$6 = \frac{150}{1 + Ae^{0}}$$
$$1 + A = 25$$
$$A = 24$$

Formula: $P = \dfrac{150}{1 + 24e^{-0.225t}}$

(b)
$$100 = \frac{150}{1 + 24e^{-0.225t}}$$
$$1 + 24e^{-0.225t} = \frac{3}{2}$$
$$24e^{-0.225t} = \frac{1}{2}$$
$$e^{-0.225t} = \frac{1}{48}$$
$$-0.225t = -\ln 48$$
$$t = \frac{\ln 48}{0.225} \approx 17.21 \text{ weeks}$$
$$125 = \frac{150}{1 + 24e^{-0.225t}}$$
$$1 + 24e^{-0.225t} = \frac{6}{5}$$
$$24e^{-0.225t} = \frac{1}{5}$$
$$e^{-0.225t} = \frac{1}{120}$$
$$-0.225t = -\ln 120$$
$$t = \frac{\ln 120}{0.225} \approx 21.28$$

It will take about 17.21 weeks to reach 100 guppies, and about 21.28 weeks to reach 125 guppies.

34. **(a)** $\dfrac{dP}{dt} = 0.0004P(250 - P) = kP(M - P)$

Thus, $k = 0.0004$ and $M = 250$.

$P = \dfrac{M}{1 + Ae^{-Mkt}} = \dfrac{250}{1 + Ae^{-0.1t}}$

Initial condition: $P(0) = 28$, where $t = 0$ represents the year 1970.

$$28 = \frac{250}{1+Ae^0}$$
$$28(1+A) = 250$$
$$A = \frac{250}{28} - 1 = \frac{111}{14} \approx 7.9286$$

Formula:

$$P(t) = \frac{250}{1+111e^{-0.1t}/14}, \text{ or approximately}$$
$$P(t) = \frac{250}{1+7.9286e^{-0.1t}}$$

(b) The population $P(t)$ will round to 250 when $P(t) \geq 249.5$.

$$249.5 = \frac{250}{1+111e^{-0.1t}/14}$$
$$249.5\left(1+\frac{111e^{-0.1t}}{14}\right) = 250$$
$$\frac{(249.5)(111e^{-0.1t})}{14} = 0.5$$
$$e^{-0.1t} = \frac{14}{55{,}389}$$
$$-0.1t = \ln\frac{14}{55{,}389}$$
$$t = 10(\ln 55{,}389 - \ln 14)$$
$$\approx 82.8$$

It will take about 83 years.

35.
$$\frac{dP}{dt} = kP(M-P)$$
$$\int \frac{dP}{P(M-P)} = \int k\,dt$$
$$\frac{Q}{P} + \frac{R}{M-P} = \frac{1}{P(M-P)}$$
$$Q(M-P) + RP = 1$$
$$P = 0,\ MQ = 1$$
$$Q = \frac{1}{M}$$
$$P = M,\ MR = 1$$
$$R = \frac{1}{M}$$
$$\int\left(\frac{\frac{1}{M}}{P} + \frac{\frac{1}{M}}{(M-P)}\right)dP = kt + C$$
$$\int\left(\frac{1}{P} + \frac{1}{(M-P)}\right)dP = Mkt + C$$
$$\ln\left(\frac{M-P}{P}\right) = -Mkt - C$$
$$\frac{M}{P} - 1 = e^{-Mkt}e^{-C}$$
$$\frac{M}{P} = 1 + e^{-Mkt}A$$
$$P = \frac{M}{1+Ae^{-Mkt}}$$

36. (a)
$$\frac{dP}{dt} = k(M-P)$$
$$\int \frac{dP}{M-P} = \int k\,dt$$
$$-\ln(M-P) = kt + C$$
$$M - P = e^{-kt}\ e^{-c}$$

Let $e^{-c} = A$ then $P = M - Ae^{-kt}$.

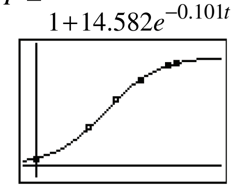
(b) $\lim_{t\to\infty} P(t) = M - Ae^{-k\infty} = M$

(c) When $t = 0$.

(d) This curve has no inflection point. If the initial population is greater than M, the curve is always concave up and approaches $y = M$ asymptotically from above. If the initial population is smaller than M, the curve is always concave down and approaches $y = M$ asymptotically from below.

37. (a) The regression equation is

$$P = \frac{232739.9}{1+14.582e^{-0.101t}}$$



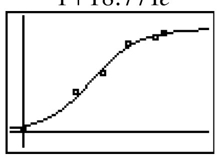
[–5, 70] by [–24000, 260000]

(b)
$$\lim_{t\to\infty}\frac{232{,}739.9}{1+14.582e^{-0.101t}} = \frac{232{,}739.9}{1+0}$$
$$\approx 232{,}740 \text{ people.}$$

(c)
$$225{,}000 = \frac{232{,}739.9}{1+14.582e^{-0.101t}}$$
$$14.582e^{-0.101t} = \frac{232{,}739.9}{225{,}000} - 1$$
$$e^{-0.101t} = 0.0024$$
$$t = \frac{-6.05}{-0.101} \approx 60 \text{ or in 2010.}$$

(d)
$$\frac{dP}{dt} = kP(M-P)$$
$$= (4.352\times 10^{-7})P(232{,}739.9 - P).$$

38. (a) The regression equation is
$$P = \frac{458791.8}{1+18.771e^{-0.113t}}$$



(b)
$$\lim_{t\to\infty} \frac{458{,}791.8}{1+18.771e^{-0.113(t)}} = \frac{458{,}791.8}{1+0} \approx 458{,}792$$

(c)
$$450{,}000 = \frac{458{,}791.8}{1+18.771e^{-0.113t}}$$
$$18.771e^{-0.113t} = \frac{458791.8}{450{,}000} - 1$$
$$e^{-0.113t} = 0.001$$
$$t = \frac{-6.87}{-0.113} = 60 \text{ or in 2010.}$$

(d)
$$\frac{dP}{dt} = kP(M-P)$$
$$= (2.4626\times 10^{-7})P(458791.8 - P)$$

39. False; it does look exponential, but it resembles the solution to
$$\frac{dP}{dt} = kP(100-10) = (90k)P.$$

40. True; the graph will be a logistic curve with $\lim_{t\to\infty} P(t) = 100$ and $\lim_{t\to-\infty} P(t) = 0$.

41. D; $\frac{600}{2} = 300.$

42. B; $M = 0.9$, so at most 90% of the population will be infected. The remaining 10% will not be infected.

43. D; $\int_2^3 \frac{3}{(x-1)(x+2)}\,dx$
$$\frac{A}{x-1} + \frac{B}{x+2} = \frac{3}{(x-1)(x+2)}$$
$$A(x+2) + B(x-1) = 3$$
$$x = -2,\ B(-2-1) = 3$$
$$-3B = 3$$
$$B = -1$$
$$x = 1,\ A(1+2) = 3$$
$$3A = 3$$
$$A = 1$$
$$\int\left(\frac{1}{x-1} + \frac{-1}{x+2}\right)dx = \ln\left(\frac{x-1}{x+2}\right)\Bigg|_2^3 = \ln\left(\frac{8}{5}\right)$$

44. B

45. (a) Note that $k > 0$ and $M > 0$, so the sign of $\frac{dP}{dt}$ is the same as the sign of $(M-P)(P-m)$. For $m < P < M$, both $M - P$ and $P - m$ are positive, so the product is positive. For $P < m$ or $P > M$, the expressions $M - P$ and $P - m$ have opposite signs, so the product is negative.

(b) $\dfrac{dP}{dt} = \dfrac{k}{M}(M-P)(P-m)$

$$\frac{dP}{dt} = \frac{k}{1200}(1200-P)(P-100)$$

$$\frac{1200}{(1200-P)(P-100)}\frac{dP}{dt} = k$$

$$\frac{1100}{(1200-P)(P-100)}\frac{dP}{dt} = \frac{11}{12}k$$

$$\frac{(P-100)+(1200-P)}{(1200-P)(P-100)}\frac{dP}{dt} = \frac{11}{12}k$$

$$\left(\frac{1}{1200-P} + \frac{1}{P-100}\right)\frac{dP}{dt} = \frac{11}{12}k$$

$$\int\left(\frac{1}{1200-P} + \frac{1}{P-100}\right)dP = \int\frac{11}{12}k\ dt$$

$$-\ln|1200-P| + \ln|P-100| = \frac{11}{12}kt + C$$

$$\ln\left|\frac{P-100}{1200-P}\right| = \frac{11}{12}kt + C$$

$$\frac{P-100}{1200-P} = \pm e^{C}e^{11kt/12}$$

$$\frac{P-100}{1200-P} = Ae^{11kt/12}$$

$$P - 100 = 1200Ae^{11kt/12} - APe^{11kt/12}$$

$$P(1 + Ae^{11kt/12}) = 1200Ae^{11kt/12} + 100$$

$$P = \frac{1200Ae^{11kt/12} + 100}{1 + Ae^{11kt/12}}$$

(c) $300 = \dfrac{1200Ae^{0} + 100}{1 + Ae^{0}}$

$$300(1 + A) = 1200A + 100$$

$$300 - 100 = 1200A - 300A$$

$$200 = 900A$$

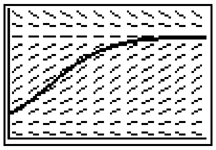
$$A = \frac{2}{9}$$

$$P(t) = \frac{1200\left(\frac{2}{9}\right)e^{11kt/12} + 100}{1 + \left(\frac{2}{9}\right)e^{11kt/12}}$$

$$P(t) = \frac{1200(2)e^{11kt/12} + 100(9)}{9 + 2e^{11kt/12}}$$

$$P(t) = \frac{300(8e^{11kt/12} + 3)}{9 + 2e^{11kt/12}}$$

(d)



[0, 75] by [0, 1500]

Note that the slope field is given by

$$\frac{dP}{dt} = \frac{0.1}{1200}(1200 - P)(P - 100).$$

(e)

$$\frac{dP}{dt} = \frac{k}{M}(M - P)(P - m)$$

$$\frac{M}{(M-P)(P-m)}\frac{dP}{dt} = k$$

$$\frac{M}{M-m}\frac{M-m}{(M-P)(P-m)}\frac{dP}{dt} = k$$

$$\frac{(P-m)+(M-P)}{(M-P)(P-m)}\frac{dP}{dt} = \frac{M-m}{M}k$$

$$\left(\frac{1}{M-P} + \frac{1}{P-m}\right)\frac{dP}{dt} = \frac{M-m}{M}k$$

$$\int\left(\frac{1}{M-P} + \frac{1}{P-m}\right)dP = \int\frac{M-m}{M}k\,dt$$

$$-\ln|M-P| + \ln|P-m| = \frac{M-m}{M}kt + C$$

$$\ln\left|\frac{P-m}{M-P}\right| = \frac{M-m}{M}kt + C$$

$$\frac{P-m}{M-P} = \pm e^C e^{(M-m)kt/M}$$

$$\frac{P-m}{M-P} = Ae^{(M-m)kt/M}$$

$$P - m = (M-P)Ae^{(M-m)kt/M}$$

$$P(1 + Ae^{(M-m)kt/M}) = AMe^{(M-m)kt/M} + m$$

$$P = \frac{AMe^{(M-m)kt/M} + m}{1 + Ae^{(M-m)kt/M}}$$

$$P(0) = \frac{AMe^0 + m}{1 + Ae^0} = \frac{AM + m}{1 + A}$$

$$P(0)(1 + A) = AM + m$$

$$A(P(0) - M) = m - P(0)$$

$$A = \frac{m - P(0)}{P(0) - M} = \frac{P(0) - m}{M - P(0)}$$

Therefore, the solution to the differential equation is

$$P = \frac{AMe^{(M-m)kt/M} + m}{1 + Ae^{(M-m)kt/M}} \text{ where } A = \frac{P(0) - m}{M - P(0)}.$$

46. **(a)** Let $u = \frac{x}{a}$; then $x = au$, $dx = a\,du$

$$\int \frac{dx}{a^2 + x^2} = \int \frac{a\,du}{a^2 + a^2u^2}$$
$$= \frac{a}{a^2}\int \frac{du}{1+u^2}$$
$$= \frac{1}{a}\tan^{-1}(u) + C$$
$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

(b) Let $u = \frac{x}{a}$; then $x = au$, $dx = a\,du$

$$\int \frac{dx}{a^2 - x^2} = \int \frac{a\,du}{a^2 - a^2u^2}$$
$$= \frac{a}{a^2}\int \frac{du}{1-u^2}$$
$$= -\frac{1}{a}\int \frac{du}{u^2-1}$$
$$= -\frac{1}{a}\int \frac{1}{(u+1)(u-1)}du$$
$$\frac{A}{u+1} + \frac{B}{u-1} = \frac{1}{(u+1)(u-1)}$$
$$A(u-1) + B(u+1) = 1$$
$u = -1$, $A(-2) = 1$
$$A = -\frac{1}{2}$$
$u = 1$, $B(2) = 1$
$$B = \frac{1}{2}$$
$$-\frac{1}{a}\int\left(\frac{-\frac{1}{2}}{u+1} + \frac{\frac{1}{2}}{u-1}\right)du$$
$$= \frac{1}{2a}\int\left(\frac{1}{u+1} - \frac{1}{u-1}\right)du$$
$$= \frac{1}{2a}\ln\left|\frac{u+1}{u-1}\right| + C$$
$$= \frac{1}{2a}\ln\left|\frac{\frac{x}{a}+1}{\frac{x}{a}-1}\right| + C$$
$$= \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$$

(c) Let $u = a + x$, $du = dx$

$$\int \frac{dx}{(a+x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{x+a} + C$$

47. **(a)** $\int \frac{5x}{(x+3)^2}dx$

$$\frac{A}{x+3} + \frac{B}{(x+3)^2} = \frac{5x}{(x+3)^2}$$
$$A(x+3) + B = 5x$$
$x = 3$, $B = -15$
$x = 0$, $A(x+3) - 15 = 5x$
$A(3) - 15 = 0$
$A = 5$
$$\int\left(\frac{5}{x+3} - \frac{15}{(x+3)^2}\right)dx$$
$$= 5\ln|x+3| + \frac{15}{x+3} + C$$

(b) $\int \frac{5x}{(x+3)^3}dx$

$$\frac{A}{(x+3)} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} = \frac{5x}{(x+3)^3}$$
$$A(x+3)^2 + B(x+3) + C = 5x$$
$x = -3$, $C = 15$
$$A(x+3)^2 + B(x+3) - 15 = 5x$$
$x = 0$, $9A + 3B - 15 = 0$, $B = 5 + 3A$
$x = 1$, $16A + 4B - 15 = 5$, $B = 5 + 4A$
$5 + 3A + 5 + 4A$
$A = 0$
$B = 5 + 3(0) = 5$
$$\int\left(\frac{5}{(x+3)^2} - \frac{15}{(x+3)^3}\right)dx$$
$$= -\frac{5}{x+3} + \frac{15}{2(x+3)^2} + C$$

48. **(a)** This is true since

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$
$$= \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$

(b) $A(x-1)^2 + B(x-1) + C = x^2 + 3x + 5$

$x = 0$, $A - B + C = 5$
$x = 1$, $C = 9$
$x = 2$, $A + B + C = 15$
Then
$$\begin{array}{r} A - B = -4 \\ \underline{A + B = 6} \\ 2A = 2 \\ A = 1 \end{array}$$
$B = 5$

(c) $\int \left(\frac{1}{x-1} + \frac{5}{(x-1)^2} + \frac{9}{(x-1)^3} \right) dx$

$$= \ln|x-1| - \frac{5}{x-1} - \frac{9}{2(x-1)^2} + C$$

Quick Quiz Sections 7.4 and 7.5

1. C; $y = y_0 e^{kt}$

$t = 1,\ 2 = y_0 e^k$

$t = 5,\ \ 3 = y_0 e^{5k}$

$$\frac{3}{2} = \frac{y_0 e^{5k}}{y_0 e^k} = e^{4k}$$

$$k = \frac{\ln\left(\frac{3}{2}\right)}{4}$$

$$y_0 = 2e^{-\frac{\ln(3/2)}{4}} = 1.807$$

$$t = 8,\ y = 1.807 e^{\left(\frac{\ln(3/2)}{4} \cdot 8\right)} = 4.066$$

2. C; $F(x) = \int_a^x \cos(t^2)dt$

$F(1) = 0 \quad \Rightarrow \quad a = 1$

$F(x) = \int_1^x \cos(t^2)dt$

$F(5) = \int_1^5 \cos(t^2)dt = -0.293$

[Use NINT $(\cos(x^2), x, 1, 5)$ to evaluate the integral.]

3. A; $\int \frac{dx}{(x-1)(x+3)}$

$$\frac{A}{x-1} + \frac{B}{x+3} = \frac{1}{(x-1)(x+3)}$$

$A(x+3) + B(x-1) = 1$

$x = -3,\ -4B = 1$

$$B = -\frac{1}{4}$$

$x = 1,\ 4A = 1$

$$A = \frac{1}{4}$$

$$\int \left(\frac{\frac{1}{4}}{x-1} + \frac{-\frac{1}{4}}{x+3} \right) dx = \frac{1}{4} \ln\left|\frac{x-1}{x+3}\right| + C$$

4. $\frac{dP}{dt} = \frac{P}{5}\left(\frac{10-P}{10}\right)$

$$\int \frac{dP}{P(10-P)} = \int \frac{1}{50} dt$$

$$\frac{A}{P} + \frac{B}{10-P} = \frac{1}{P(10-P)}$$

$A(10-P) + BP = 1$

$P = 10,\ 10B = 1$

$B = 0.1$

$P = 0,\ 10A = 1$

$A = 0.1$

$$\int \left(\frac{0.1}{P} + \frac{0.1}{10-P} \right) dP = \frac{1}{50} t + C$$

$$\ln\left|\frac{10-P}{P}\right| = -\frac{1}{5} t - C$$

$$P = \frac{10}{1 + e^{-1/5t} e^{-C}}$$

(a) $P(0) = 3 = \frac{10}{1 + Ae^{-1/5(0)}}$

$A = 2.33$

$$\lim_{t\to\infty} P(t) = \frac{10}{1 + 2.33e^{-1/5(t)}} = 10$$

(b) $P(0) = 20 = \frac{10}{1 + Ae^{-1/5(0)}}$

$A = -0.5$

$$\lim_{t\to\infty} P(t) = \frac{10}{1 + -0.5e^{-1/5(t)}} = 10$$

(c) Separate the variables.

$$\frac{dY}{Y} = \frac{1}{5}\left(1 - \frac{t}{10}\right) dt$$

$$\ln Y = \frac{t}{5} - \frac{t^2}{100} + C_1$$

$Y = Ce^{t/5 - t^2/100}$ where $C = e^{C_1}$

$3 = Ce^0 \Rightarrow C = 3$

$Y = 3e^{t/5 - t^2/100}$

(d) $\lim_{t\to\infty} 3e^{t/5 - t^2 100} = \lim_{t\to\infty} \frac{3e^{t/5}}{e^{t^2/100}}$

$$= \lim_{t\to\infty} \frac{3e^{t/5}}{(e^{t/5})^{t/20}}$$

$$= 0$$

Chapter 7 Review Exercises (pp. 377–380)

1. $\int_0^{\pi/3} \sec^2\theta\, d\theta = \tan\theta\Big|_0^{\pi/3}$
$$= \tan\frac{\pi}{3} - \tan 0$$
$$= \sqrt{3}$$

2. $\int_1^2 \left(x + \frac{1}{x^2}\right) dx = \left[\frac{1}{2}x^2 - x^{-1}\right]_1^2$
$$= \left(\frac{1}{2}(4) - \frac{1}{2}\right) - \left(\frac{1}{2} - 1\right)$$
$$= \frac{3}{2} + \frac{1}{2}$$
$$= \frac{4}{2}$$
$$= 2$$

3. Let $u = 2x + 1$, $du = 2\,dx$, $\frac{1}{2}du = dx$

$$\int_0^1 \frac{36}{(2x+1)^3}\,dx = 18\int_1^3 \frac{1}{u^3}\,du$$
$$= 18\left(-\frac{1}{2}\right)u^{-2}\Big|_1^3$$
$$= -9\left(\frac{1}{9} - 1\right)$$
$$= -9\left(-\frac{8}{9}\right)$$
$$= 8$$

4. Let $u = 1 - x^2$, $du = -2x\,dx$, $-du = 2x\,dx$

$$\int_{-1}^1 2x\sin(1-x^2)\,dx = -\int_0^0 \sin u\,du = 0$$

5. Let $u = \sin x$, $du = \cos x\,dx$

$$\int_0^{\pi/2} 5\sin^{3/2} x\cos x\,dx$$
$$= \int_0^1 5u^{3/2}\,du$$
$$= 5\cdot\frac{2}{5}u^{5/2}\Big|_0^1$$
$$= 2(1-0)$$
$$= 2$$

6. $\int_{1/2}^4 \frac{x^2+3x}{x}\,dx = \int_{1/2}^4 (x+3)\,dx\ (x \ne 0)$
$$= \left(\frac{1}{2}x^2 + 3x\right)\Big|_{1/2}^4$$
$$= \left(\frac{1}{2}(16) + 3(4)\right) - \left(\frac{1}{2}\left(\frac{1}{4}\right) + \frac{3}{2}\right)$$
$$= 20 - \left(\frac{1}{8} + \frac{12}{8}\right)$$
$$= 20 - \frac{13}{8}$$
$$= \frac{147}{8}$$

7. Let $u = \tan x$, $du = \sec^2 x\,dx$

$$\int_0^{\pi/4} e^{\tan x}\sec^2 x\,dx = \int_0^1 e^u\,du$$
$$= e^u\Big|_0^1$$
$$= e^1 - e^0$$
$$= e - 1$$

8. Let $u = \ln r$, $du = \frac{1}{r}dr$

$$\int_1^e \frac{\sqrt{\ln r}}{r}\,dr = \int_0^1 u^{1/2}\,du$$
$$= \frac{2}{3}u^{3/2}\Big|_0^1$$
$$= \frac{2}{3}(1-0)$$
$$= \frac{2}{3}$$

9. $\int_0^1 \frac{x}{x^2+5x+6}\,dx$

$$\frac{x}{(x+3)(x+2)}$$
$$\frac{A}{x+3} + \frac{B}{x+2} = \frac{x}{(x+3)(x+2)}$$
$$A(x+2) + B(x+3) = x$$
$$x = -2,\ B(-2+3) = -2$$
$$B = -2$$

$$x = -3,\ A(-3+2) = -3$$
$$-A = -3$$
$$A = 3$$
$$\int \frac{3}{x+3} + \frac{-2}{x+2}\,dx = \ln(x+3)^3 - \ln(x+2)^2\Big|_0^1$$
$$= \ln\left(\frac{256}{243}\right)$$

10. $\int_1^2 \frac{2x+6}{x^2-3x}\,dx$

$$\frac{2x+6}{x(x-3)}$$
$$\frac{A}{x} + \frac{B}{x-3} = \frac{2x+6}{x(x-3)}$$
$$A(x-3) + Bx = 2x+6$$
$$x = 3,\ 3B = 2(3)+6$$
$$B = 4$$
$$x = 0,\ A(0-3) = 2(0)+6$$
$$-3A = 6$$
$$A = -2$$
$$\int_1^2 \frac{-2}{x} + \frac{4}{x-3}\,dx = -2\ln x + 4\ln(x-3)\Big|_1^2$$
$$= -6\ln 2$$

11. Let $u = 2 - \sin x$, $du = -\cos x\, dx$, $-du = \cos x\, dx$

$$\int \frac{\cos x}{2-\sin x}\,dx = -\int \frac{1}{u}\,du$$
$$= -\ln|u| + C$$
$$= -\ln|2-\sin x| + C$$

12. Let $u = 3x + 4$, $\quad du = 3\,dx$

$$\frac{1}{3}du = dx$$
$$\int \frac{dx}{\sqrt[3]{3x+4}} = \frac{1}{3}\int u^{-1/3}\,du$$
$$= \frac{1}{3}\cdot\frac{3}{2}u^{2/3} + C$$
$$= \frac{1}{2}(3x+4)^{2/3} + C$$

13. Let $u = t^2 + 5$, $\quad du = 2t\,dt$

$$\frac{1}{2}du = t\,dt$$
$$\int \frac{t\,dt}{t^2+5} = \frac{1}{2}\int \frac{1}{u}\,du$$
$$= \frac{1}{2}\ln|u| + C$$
$$= \frac{1}{2}\ln\left|t^2+5\right| + C$$
$$= \frac{1}{2}\ln(t^2+5) + C$$

14. Let $u = \frac{1}{\theta}$, $\ du = -\frac{1}{\theta^2}\,d\theta$

$$\int \frac{1}{\theta^2}\sec\frac{1}{\theta}\tan\frac{1}{\theta}\,d\theta = -\int \sec u \tan u\,du$$
$$= -\sec u + C$$
$$= -\sec\frac{1}{\theta} + C$$

15. Let $u = \ln y$, $du = \frac{1}{y}dy$

$$\int \frac{\tan(\ln y)}{y}\,dy = \int \tan u\,du$$
$$= \int \frac{\sin u}{\cos u}\,du$$

Let $w = \cos u$
$dw = -\sin u\,du$

$$= -\int \frac{1}{w}\,dw$$
$$= -\ln|w| + C$$
$$= -\ln|\cos u| + C$$
$$= -\ln|\cos(\ln y)| + C$$

16. Let $u = e^x$, $\ du = e^x dx$

$$\int e^x \sec(e^x)\,dx = \int \sec u\,du$$
$$= \ln|\sec u + \tan u| + C$$
$$= \ln\left|\sec(e^x) + \tan(e^x)\right| + C$$

17. Let $u = \ln x$, $du = \frac{1}{x}dx$

$$\int \frac{dx}{x\ln x} = \int \frac{1}{u}\,du$$
$$= \ln|u| + C$$
$$= \ln|\ln x| + C$$

18. $\int \frac{dt}{t\sqrt{t}} = \int \frac{dt}{t^{3/2}}$
$= \int t^{-3/2}\,dt$
$= -2t^{-1/2} + C$
$= -\frac{2}{\sqrt{t}} + C$

19. Use tabular integration with $f(x) = x^3$ and $g(x) = \cos x$.

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	(+)	$\cos x$
$3x^2$	(−)	$\sin x$
$6x$	(+)	$-\cos x$
6	(−)	$-\sin x$
0		$\cos x$

$\int x^3 \cos x\,dx$
$= x^3 \sin x + 3x^2 \cos x - 6x\sin x - 6\cos x + C$

20. Let $u = \ln x \qquad dv = x^4\,dx$
$du = \frac{1}{x}dx \qquad v = \frac{1}{5}x^5$

$$\int x^4 \ln x\,dx = \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^5\left(\frac{1}{x}\right)dx$$
$$= \frac{1}{5}x^5 \ln x - \frac{1}{5}\int x^4\,dx$$
$$= \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + C$$

21. Let $u = e^{3x} \qquad dv = \sin x\,dx$
$du = 3e^{3x}\,dx \qquad v = -\cos x$
$\int e^{3x} \sin x\,dx = -e^{3x}\cos x + \int 3\cos x\, e^{3x}\,dx$
Integrate by parts again
Let $u = 3e^{3x} \qquad dv = \cos x\,dx$
$du = 9e^{3x}\,dx \qquad v = \sin x$
$\int e^{3x} \sin x\,dx$
$= -e^{3x}\cos x + 3e^{3x}\sin x - \int 9e^{3x}\sin x\,dx$
$10\int e^{3x}\sin x\,dx = -e^{3x}\cos x + 3e^{3x}\sin x + C$
$\int e^{3x}\sin x\,dx$
$= \frac{1}{10}[-e^{3x}\cos x + 3e^{3x}\sin x] + C$
$= \left(\frac{3\sin x}{10} - \frac{\cos x}{10}\right)e^{3x} + C$

22. Let $u = x^2 \qquad dv = e^{-3x}\,dx$
$du = 2x\,dx \qquad v = -\frac{1}{3}e^{-3x}$
$\int x^2 e^{-3x}\,dx$
$= -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\int e^{-3x}x\,dx$
Let $u = x \qquad dv = e^{-3x}\,dx$
$du = dx \qquad v = -\frac{1}{3}e^{-3x}$
$= -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\left[-\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}\,dx\right]$
$= -\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} + \frac{2}{9}\int e^{-3x}\,dx$
$= -\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C$
$= \left(-\frac{x^2}{3} - \frac{2x}{9} - \frac{2}{27}\right)e^{-3x} + C$

23. $\int \frac{25}{x^2 - 25}dx = \int \frac{25}{(x+5)(x-5)}dx$
$\frac{A}{x+5} + \frac{B}{x-5} = \frac{25}{(x+5)(x-5)}$
$A(x-5) + B(x+5) = 25$
$x = 5,\ B(5+5) = 25$
$10B = 25$
$B = \frac{5}{2}$
$x = -5,\ A(-5-5) = 25$
$-10A = 25$
$A = -\frac{5}{2}$

$$\int\left(\frac{-\frac{5}{2}}{x+5} + \frac{\frac{5}{2}}{x-5}\right)dx = \frac{5}{2}\ln\left|\frac{x-5}{x+5}\right| + C$$

24. $$\int \frac{5x+2}{2x^2+x-1}dx = \int \frac{5x+2}{(2x-1)(x+1)}dx$$

$$\frac{A}{2x-1}+\frac{B}{x+1}=\frac{5x+2}{(2x-1)(x+1)}$$

$$A(x+1)+B(2x-1)=5x+2$$

$$x=-1,\ B(2(-1)-1)=5(-1)+2$$

$$-3B=-3$$

$$B=1$$

$$x=\frac{1}{2},\quad A\left(\frac{1}{2}+1\right)=5\left(\frac{1}{2}\right)+2$$

$$\frac{3}{2}A=\frac{9}{2}$$

$$A=3$$

$$\int\left(\frac{3}{2x-1}+\frac{1}{x+1}\right)dx$$

$$=\frac{3}{2}\ln|2x-1|+\ln|x+1|$$

$$=\frac{1}{2}\ln\left|(2x-1)^3(x+1)^2\right|+C$$

25. $$\frac{dy}{dx}=1+x+\frac{x^2}{2}$$

$$dy=\left(1+x+\frac{x^2}{2}\right)dx$$

$$\int dy=\int\left(1+x+\frac{x^2}{2}\right)dx$$

$$y=x+\frac{1}{2}x^2+\frac{1}{6}x^3+C$$

$$y(0)=C=1$$

$$y=\frac{x^3}{6}+\frac{x^2}{2}+x+1$$

26. $$\frac{dy}{dx}=\left(x+\frac{1}{x}\right)^2$$

$$dy=\left(x+\frac{1}{x}\right)^2 dx$$

$$\int dy=\int\left(x+\frac{1}{x}\right)^2 dx$$

$$y=\int\left(x^2+2+\frac{1}{x^2}\right)dx$$

$$y=\frac{1}{3}x^3+2x-x^{-1}+C$$

$$y(1)=\frac{1}{3}+2-1+C=1$$

$$\frac{4}{3}+C=1$$

$$C=-\frac{1}{3}$$

$$y=\frac{x^3}{3}+2x-\frac{1}{x}-\frac{1}{3}$$

27. $$\frac{dy}{dt}=\frac{1}{t+4}$$

$$dy=\frac{1}{t+4}dt$$

$$\int dy=\int\frac{1}{t+4}dt$$

$$y=\ln|t+4|+C$$

$$y(-3)=\ln(1)+C=2$$

$$C=2$$

$$y=\ln(t+4)+2$$

28. $$\frac{dy}{d\theta}=\csc 2\theta\cot 2\theta$$

$$dy=\csc 2\theta\cot 2\theta\, d\theta$$

$$\int dy=\int\csc 2\theta\cot 2\theta\, d\theta$$

$$y=-\frac{1}{2}\csc 2\theta+C$$

$$y\left(\frac{\pi}{4}\right)=-\frac{1}{2}+C=1$$

$$C=\frac{3}{2}$$

$$y=-\frac{1}{2}\csc 2\theta+\frac{3}{2}$$

29. $$\frac{d(y')}{dx}=2x-\frac{1}{x^2}$$
$$d(y')=\left(2x-\frac{1}{x^2}\right)dx$$
$$\int d(y')=\int\left(2x-\frac{1}{x^2}\right)dx$$
$$y'=x^2+x^{-1}+C$$
$$y'(1)=2+C=1$$
$$C=-1$$
$$y'=x^2+x^{-1}-1$$
$$\int dy=\int(x^2+x^{-1}-1)\,dx$$
$$y=\frac{1}{3}x^3+\ln x-x+C$$
$$y'=x^2+x^{-1}-1$$
$$\int dy=\int(x^2+x^{-1}-1)\,dx$$
$$y=\frac{1}{3}x^3+\ln x-x+C$$
$$y(1)=\frac{1}{3}+0-1+C=0$$
$$-\frac{2}{3}+C=0$$
$$C=\frac{2}{3}$$
$$y=\frac{x^3}{3}+\ln x-x+\frac{2}{3}$$

30. $$\frac{d(r'')}{dt}=-\cos t$$
$$d(r'')=-\cos t\,dt$$
$$\int d(r'')=\int-\cos t\,dt$$
$$r''=-\sin t+C$$
$$r''(0)=C=-1$$
$$r''=-\sin t-1$$
$$\int d(r')=\int(-\sin t-1)\,dt$$
$$r'=\cos t-t+C$$
$$r'(0)=1+C=-1$$
$$C=-2$$
$$r'=\cos t-t-2$$
$$\int dr=\int(\cos t-t-2)\,dt$$
$$r=\sin t-\frac{t^2}{2}-2t+C$$
$$r(0)=C=-1$$
$$r=\sin t-\frac{t^2}{2}-2t-1$$

31. $$\frac{dy}{dx}=y+2$$
$$\frac{dy}{y+2}=dx$$
$$\int\frac{dy}{y+2}=\int dx$$
$$\ln|y+2|=x+C$$
$$y+2=Ce^x$$
$$y=Ce^x-2$$
$$y(0)=C-2=2$$
$$C=4$$
$$y=4e^x-2$$

32. $$\frac{dy}{dx}=(2x+1)(y+1)$$
$$\frac{dy}{y+1}=(2x+1)\,dx$$
$$\int\frac{dy}{y+1}=\int(2x+1)\,dx$$
$$\ln|y+1|=x^2+x+C$$
$$y+1=Ce^{x^2+x}$$
$$y=Ce^{x^2+x}-1$$
$$y(-1)=C-1=1$$
$$C=2$$
$$y=2e^{x^2+x}-1$$

33. $$\frac{dy}{dt}=y(1-y)$$
$$\frac{dy}{y(1-y)}=dt$$
$$\frac{A}{y}+\frac{B}{1-y}=\frac{1}{y(1-y)}$$
$$A(1-y)+By=1$$
$$y=1,\ B=1$$
$$y=0,\ A=1$$
$$\int\left(\frac{1}{y}+\frac{1}{1-y}\right)dy=\int dt$$
$$\ln|y|-\ln|1-y|=t+C$$
$$\ln\left|\frac{1-y}{y}\right|=-t-C$$
$$\frac{1-y}{y}=e^{-t}e^{-c}$$
$$\frac{1}{y}-1=e^{-t}e^{-c}$$
$$y=\frac{1}{1+Ae^{-t}}$$

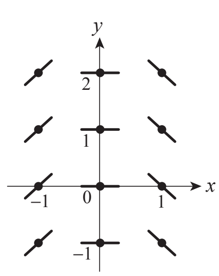
$$y(0) = 0.1 = \frac{1}{1 + Ae^{-(0)}}$$
$$A = 9$$
$$y = \frac{1}{1 + 9\,e^{-t}}$$

34. $$\frac{dy}{dx} = 0.001y\,(100 - y)$$
$$\frac{dy}{0.001y\,(100 - y)} = dx$$
$$\frac{A}{0.001\,y} + \frac{B}{100 - y} = \frac{1}{0.001y(100 - y)}$$
$$A(100 - y) + B(0.001y) = 1$$
$$y = 100,\ B(0.1) = 1$$
$$B = 10$$
$$y = 0,\ 100A = 1$$
$$A = 0.01$$
$$\int\left(\frac{0.01}{0.001\,y} + \frac{10}{100 - y}\right)dy = x + C$$
$$\int\left(\frac{0.001}{0.001\,y} + \frac{1}{100 - y}\right)dy = 0.1x + C$$
$$\ln y - \ln|100 - y| = 0.1x + C$$
$$\ln\left|\frac{100 - y}{y}\right| = -0.1x - C$$
$$\frac{100}{y} - 1 = e^{-0.1x}e^{-c}$$
$$y = \frac{100}{1 + Ae^{-0.1\,x}}$$
$$y(0) = 5 = \frac{100}{1 + Ae^{-0.1(0)}}$$
$$A = 19$$
$$y = \frac{100}{1 + 19e^{-0.1x}}$$

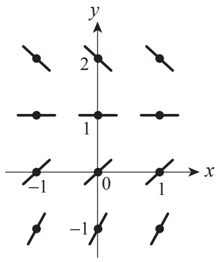
35. $y = \int_4^x \sin^3 t\; dt + 5$

36. $y = \int_1^x \sqrt{1 + t^4}\; dt + 2$

37.



38.



39. Graph (b). Slope lines are vertical for points on the line $y = -x$.

40. Graph (d). Slope lines are vertical for points on the line $y = x$.

41. Graph (c). Slope lines are horizontal for points on the x- and y-axes. Slopes are positive in Quadrants I and III. Slopes are negative in Quadrants II and IV.

42. Graph (a). Slope lines are horizontal for points on the x- and y-axes. Slopes are positive in Quadrants II and IV; negative in Quadrants I and III.

43.

(x, y)	$\frac{dy}{dx} = x + y - 1$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(1, 1)	1.0	0.1	0.1	(1.1, 1.1)
(1.1, 1.1)	1.2	0.1	0.12	(1.2, 1.22)
(1.2, 1.22)	1.42	0.1	0.142	(1.3, 1.362

$y = 1.362$

44.

(x, y)	$\frac{dy}{dx} = x - y$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$(x + \Delta x, y + \Delta y)$
(1, 2)	−1.0	−0.1	0.1	(0.9, 2.1)
(0.9, 2.1)	−1.2	−0.1	0.12	(0.8, 2.22)
(0.8, 2.22)	−1.42	−0.1	0.142	(0.7, 2.362

$y = 2.362$

45. We seek the graph of a function whose derivative is $\frac{\sin x}{x}$. Graph (b) is increasing on $[-\pi, \pi]$, where $\frac{\sin x}{x}$ is positive, and oscillates slightly outside of this interval. This is the correct choice, and this can be verified by graphing NINT $\left(\frac{\sin x}{x}, x, 0, x\right)$.

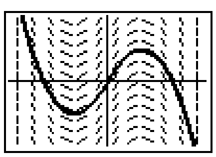
46. We seek the graph of a function whose derivative is e^{-x^2}. Since $e^{-x^2} > 0$ for all x, the desired graph is increasing for all x. Thus, the only possibility is graph (d), and we may verify that this is correct by graphing NINT $(e^{-x^2}, x, 0, x)$.

47. (iv) The given graph looks the graph of $y = x^2$, which satisfies $\frac{dy}{dx} = 2x$ and $y(1) = 1$.

48. Yes, $\frac{d^2y}{dx^2}=0$, so $\frac{dy}{dx}=C$. Since $y'(0)=1$, $\frac{dy}{dx}=1$. Then $y=\int 1\,dx=x+C$. Since $y(0)=0$, $C=0$.
$y=x$ is a solution.

49. **(a)** $\frac{dv}{dt}=2+6t$
$$\int dv=\int(2+6t)\,dt$$
$$v=2t+3t^2+C$$
Initial condition: $v=4$ when $t=0$
$$4=0+C$$
$$4=C$$
$$v=2t+3t^2+4$$

(b) $\int_0^1 v(t)\,dt=\int_0^1(2t+3t^2+4)\,dt$
$$=\Big[t^2+t^3+4t\Big]_0^1$$
$$=6-0$$
$$=6$$
The particle moves 6 m.

50. 

[–10, 10] by [–10, 10]

51. **(a)** Half-life $=\frac{\ln 2}{k}$
$$2.645=\frac{\ln 2}{k}$$
$$k=\frac{\ln 2}{2.645}$$
$$\approx 0.262059$$

(b) Mean life $=\frac{1}{k}\approx 3.81593$ years

52. $T-T_s=(T_0-T_s)e^{-kt}$
$T-40=(220-40)e^{-kt}$
Use the fact that $T=180$ and $t=15$ to find k.
$$180-40=(220-40)e^{-(k)(15)}$$
$$e^{15k}=\frac{180}{140}=\frac{9}{7}$$
$$k=\frac{1}{15}\ln\frac{9}{7}$$
$$T-40=(220-40)e^{-((1/15)\ln(9/7))t}$$
$$70-40=(220-40)e^{-((1/15)\ln(9/7))t}$$
$$e^{((1/15)\ln(9/7))t}=\frac{180}{30}=6$$
$$\left(\frac{1}{15}\ln\frac{9}{7}\right)t=\ln 6$$
$$t=\frac{15\ln 6}{\ln(9/7)}\approx 107\text{ min}$$
It took a total of about 107 minutes to cool from 220°F to 70°F. Therefore, the time to cool from 180°F to 70°F was about 92 minutes.

53. $T-T_s=(T_0-T_s)e^{-kt}$
We have the system:
$$\begin{cases}39-T_s=(46-T_s)e^{-10k}\\33-T_s=(46-T_s)e^{-20k}\end{cases}$$
Thus, $\frac{39-T_s}{46-T_s}=10^{-10k}$ and $\frac{33-T_s}{46-T_s}=e^{-20k}$
Since $(e^{-10k})^2=e^{-20k}$, this means:
$$\left(\frac{39-T_s}{46-T_s}\right)^2=\frac{33-T_s}{46-T_s}$$
$$(39-T_s)^2=(33-T_s)(46-T_s)$$
$$1521-78T_s+T_s^2=1518-79T_s+T_s^2$$
$$T_s=-3$$
The refrigerator temperature was $-3°C$.

54. See Examples 3 and 5 in Section 7.4. Use the fact that the half-life of C-14 is 5700 years to find k:
$$\frac{1}{2}=e^{-k(5700)}$$
$$\ln\left(\frac{1}{2}\right)=-5700\,k$$
$$k=\frac{\ln\left(\frac{1}{2}\right)}{-5700}=\frac{\ln 2}{5700}$$
The painting contains 99.5% of its original Carbon-14.
$$0.995=e^{\left(-\frac{\ln 2}{5700}t\right)}$$
$$\ln(0.995)=-\frac{\ln 2}{5700}t$$
$$t=-\frac{5700}{\ln 2}\ln(0.995)\approx 41.2$$
The painting is about 41.2 years old.

55. Since 90% of the Carbon-14 has decayed, 10% remains. We showed in Problem 54 that, for Carbon-14, $k = \frac{\ln 2}{5700}$.

$$0.10 = e^{\left(-\frac{\ln 2}{5700}t\right)}$$
$$\ln(0.10) = -\frac{\ln 2}{5700}t$$
$$t = -\frac{5700}{\ln 2}\ln(0.10) \approx 18{,}935$$

The sample is about 18,935 years old.

56. Use $t = 1988 - 1924 = 64$ years.

$$250\, e^{r\cdot 64} = 7500$$
$$e^{64r} = 30$$
$$64r = \ln 30$$
$$r = \frac{\ln 30}{64} \approx 0.053$$

The rate of appreciation is about 0.053, or 5.3%.

57. $L = L_0 e^{-kx}$ where x represents the depth in feet and L_0 is the surface intensity.

When $x = 18$ ft, $L = \frac{1}{2}L_0$, so

$$\frac{1}{2}L_0 = L_0 e^{-k\cdot 18}$$
$$\frac{1}{2} = e^{-k\cdot 18}$$
$$\ln\left(\frac{1}{2}\right) = -18k$$
$$k = \frac{\ln\left(\frac{1}{2}\right)}{-18} = \frac{\ln 2}{18}$$

We want to know the depth at which

$$L = \frac{1}{10}L_0$$
$$\frac{1}{10}L_0 = L_0 e^{\left(-\frac{\ln 2}{18}x\right)}$$
$$0.1 = e^{\left(-\frac{\ln 2}{18}x\right)}$$
$$\ln(0.1) = -\frac{\ln 2}{18}x$$
$$x = -\frac{18}{\ln 2}\ln(0.1) \approx 59.8.$$

You can work without artificial light to a depth of about 59.8 feet.

58. **(a)**

$$\frac{dy}{dt} = \frac{kA}{V}(c-y)$$
$$\int \frac{dy}{c-y} = \int \frac{kA}{V}\,dt$$
$$-\ln|c-y| = \frac{kA}{V}t + C$$
$$\ln|c-y| = -\frac{kA}{V}t - C$$
$$|c-y| = e^{-(kA/V)t-C}$$
$$c-y = \pm e^{-(kA/V)t-C}$$
$$y = c \pm e^{-(kA/V)t-C}$$
$$y = c + De^{-(kA/V)t}$$

Initial condition $y = y_0$ *when* $t = 0$

$$y_0 = c + D$$
$$y_0 - c = D$$

Solution: $y = c + (y_0 - c)e^{-(kA/V)t}$

(b) $\lim\limits_{t\to\infty} y(t) = \lim\limits_{t\to\infty}[c + (y_0 - c)e^{-(kA/V)t}] = c$

59. **(a)** $P(t) = \frac{150}{1+e^{4.3-t}} = \frac{150}{1+e^{4.3}e^{-t}}$

This is $P = \frac{M}{1+Ae^{-Mkt}}$ where $M = 150$, $A = e^{4.3}$, and $k = \frac{1}{150}$. Therefore, it is a solution of the logistic differential equation.

$$\frac{dP}{dt} = kP(M-P), \text{ or}$$
$$\frac{dP}{dt} = \frac{1}{150}P(150-P).$$

The carrying capacity is 150.

(b) $P(0) = \frac{150}{1+e^{4.3}} \approx 2$

Initially there were 2 infected students.

(c)

$$\frac{150}{1+e^{4.3-t}} = 125$$
$$\frac{6}{5} = 1 + e^{4.3-t}$$
$$\frac{1}{5} = e^{4.3-t}$$
$$-\ln 5 = 4.3 - t$$
$$t = 4.3 + \ln 5 \approx 5.9 \text{ days.}$$

It took about 6 days.

60. Use the Fundamental Theorem of Calculus.

$$y' = \frac{d}{dx}\left(\int_0^x \sin t^2 dt\right) + \frac{d}{dx}\left(x^3 + x + 2\right)$$
$$= (\sin x^2) + (3x^2 + 1)$$
$$y'' = \frac{d}{dx}(\sin x^2 + 3x^2 + 1)$$
$$= (\cos x^2)(2x) + 6x$$
$$= 2x\ \cos(x^2) + 6x$$

Thus, the differential equation is satisfied.
Verify the initial conditions:

$$y'(0) = (\sin 0^2) + 3(0)^2 + 1 = 1$$
$$y(0) = \int_0^0 \sin(t^2)\ dt + 0^3 + 0 + 2 = 2$$

61.
$$\frac{dP}{dt} = 0.002P\left(1 - \frac{P}{800}\right)$$
$$\frac{dP}{dt} = 0.002P\left(\frac{800 - P}{800}\right)$$
$$\frac{800}{P(800 - P)}dP = 0.002dt$$
$$\frac{A}{P} + \frac{B}{800 - P} = \frac{800}{P(800 - P)}$$
$$A(800 - P) + BP = 800$$
$P = 0, A = 1$
$P = 800, B = 1$
$$\int\left(\frac{1}{P} + \frac{1}{800 - P}\right)dP = \int 0.002\ dt$$
$$\ln|P| - \ln|800 - P| = 0.002t + C$$
$$\ln\left|\frac{P}{800 - P}\right| = 0.002t + C$$
$$\ln\left|\frac{800 - P}{P}\right| = -0.002t - C$$
$$\left|\frac{800 - P}{P}\right| = e^{-0.002t - C}$$
$$\frac{800 - P}{P} = \pm e^{-C}e^{-0.002t}$$
$$\frac{800}{P} - 1 = Ae^{-0.002t}$$
$$P = \frac{800}{1 + Ae^{-0.002t}}$$

Initial condition: $P(0) = 50$

$$50 = \frac{800}{1 + Ae^0}$$
$$1 + A = 16$$
$$A = 15$$

Solution: $P = \dfrac{800}{1 + 15e^{-0.002t}}$

62. Method 1–Compare graph of $y_1 = x^2 \ln x$ with $y_2 = \text{NDER}\left(\dfrac{x^3 \ln x}{3} - \dfrac{x^3}{9}\right)$. The graphs should be the same. Method 2–Compare graph of $y_1 = \text{NINT}(x^2 \ln x)$ with $y_2 = \dfrac{x^3 \ln x}{3} - \dfrac{x^3}{9}$. The graphs should be the same or differ only by a vertical translation.

63. **(a)**
$$20{,}000 = 10{,}000(1.063)^t$$
$$2 = 1.063^t$$
$$\ln 2 = t \ln 1.063$$
$$t = \frac{\ln 2}{\ln 1.063} \approx 11.345$$

It will take about 11.3 years.

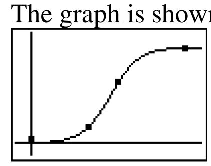
(b)
$$20{,}000 = 10{,}000e^{0.063t}$$
$$2 = e^{0.063t}$$
$$\ln 2 = 0.063t$$
$$t = \frac{\ln 2}{0.063} \approx 11.002$$

It will take about 11.0 years.

64. **(a)**
$$f'(x) = \frac{d}{dx}\int_0^x u(t)\ dt = u(x)$$
$$g'(x) = \frac{d}{dx}\int_3^x u(t)\ dt = u(x)$$

(b)
$$C = f(x) - g(x)$$
$$= \int_0^x u(t)\ dt - \int_3^x u(t)\ dt$$
$$= \int_0^x u(t)\ dt + \int_x^3 u(t)\ dt$$
$$= \int_0^3 u(t)\ dt$$

65. **(a)** The regression equation is
$$y = \frac{272286.4}{1 + 302.69e^{-0.2095t}}.$$
The graph is shown below.

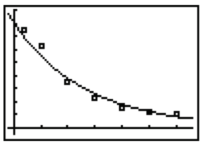


(b)
$$\lim_{t\to\infty} \frac{272{,}286.4}{1 + 302.69e^{-0.2095t}}$$
$$= \frac{272{,}286.4}{1 + 0}$$
$$= 272{,}286 \text{ people.}$$

(c) $\frac{dP}{dt} = 7.694\times10^{-7} P(272286.4 - P)$

(d) The carrying capacity drops to 267,312.6, which is below the actual 2003 population. The logistic regression is strongly affected by points at the extremes of the data, especially when there are so few data points being used. While the fit may be more dramatic for a small data set, the equation is not as reliable.

66. (a) $T = 79.961(0.9273)^t$



[–1, 33] by [–5, 90]

(b) Solving $T(t) = 40$ graphically, we obtain $t \approx 9.2\,\text{sec}$. The temperature will reach 40° after about 9.2 seconds.

(c) When the probe was removed, the temperature was about $T(0) \approx 79.76°C$.

67. (a) $\frac{1}{2}$ of the town has heard the rumor when it is spreading the fastest.

(b) $\int \frac{dy}{y(1-y)} = \int 1.2\, dt$

$$\frac{A}{y} + \frac{B}{1-y} = \frac{1}{y(1-y)}$$
$$A(1-y) + By = 1$$
$$y = 0, A = 1$$
$$y = 1, B = 1$$
$$\int\left(\frac{1}{y} + \frac{1}{1-y}\right)dy = \int 1.2\, dt$$
$$\ln\left|\frac{y}{1-y}\right| = 1.2t + C$$
$$\frac{y}{1-y} = e^c e^{1.2t}$$
$$\frac{1-y}{y} = e^{-c} e^{-1.2t}$$
$$\frac{1-y}{y} = Ae^{-1.2t}$$
$$y = \frac{1}{1+Ae^{-1.2t}}$$
$$y(0) = \frac{1}{10} = \frac{1}{1+Ae^0}$$
$$A = 9$$
$$y = \frac{1}{1+9e^{-1.2t}}$$

(c) $\frac{1}{2} = \frac{1}{1+9e^{-1.2t}}$

Solve for t to obtain

$t = \frac{5\ln 3}{3} \approx 1.83$ days.

68. (a) $\frac{dP}{dt} = k(600 - P)$. Separate the variables to obtain

$$\frac{dP}{600-P} = kdt$$
$$\frac{dP}{P-600} = -kdt$$
$$\ln|P-600| = -kt + C_1$$
$$P - 600 = Ce^{-kt}$$
$$200 - 600 = Ce^0 \Rightarrow C = -400$$
$$P - 600 = -400e^{-kt}$$
$$P(t) = 600 - 400e^{-kt}$$

(b) $500 = 600 - 400e^{-k\cdot 2}$

$$\frac{1}{4} = e^{-2k}$$
$$k = \ln 2 \approx 0.693$$

(c) $\lim_{t\to\infty}(600 - 400e^{-0.693t}) = 600$

69. (a) Separate the variables to obtain

$$\frac{dv}{v+17} = -2dt$$
$$\ln|v+17| = -2t + C_1$$
$$v + 17 = Ce^{-2t}$$
$$-47 + 17 = Ce^0 \Rightarrow C = -30$$
$$v + 17 = -30e^{-2t}$$
$$v = -30e^{-2t} - 17$$

(b) $\lim_{t\to\infty}(-30e^{-2t} - 17) = -17$ feet per second

(c) $-20 = -30e^{-2t} - 17$

$t = \frac{\ln 10}{2} \approx 1.151$ seconds